Birkhoff Coordinates of Integrable Hamiltonian systems in Asymptotic Regimes

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To my parents.

Zusammenfassung

Es werden zwei integrable Hamiltonsche Systeme untersucht: das Toda-Gitter mit periodischen Randwertbedingungen und einer grossen Anzahl Partikel und die Korteweg-de Vries (KdV) Gleichung auf $\mathbb{R}$. Im ersten Teil untersuchen wir das asymptotische Verhalten von Birkhoff Koordinaten (kartesische Wirkung- und Winkelvariablen) des Toda-Gitters in der Nähe des Gleichgewichts, falls die Anzahl Partikel N gegen Unendlich strebt. Wir zeigen, dass für geeignet gewählte Konstanten $R, R' > 0$, die der Anzahl N entsprechende Koordinatentransformation die komplexe Kugel von Radius R/N^α um den Gleichgewichtspunkt analytisch in eine Kugel mit Radius R/N^α abbildet genau dann falls $\alpha \geq 2$. Dabei werden Sobolev-analytische Normen gewählt. Als Anwendung betrachten wir das Problem der Gleichverteilung der Energie für Lösungen des Toda Gitters im Sinne von Fermi-Pasta-Ulam. Wir zeigen, dass für Anfangswerte kleiner als R/N^2, $0 < R \ll 1$, bei denen nur der erste Fourier Koeffizient nicht null ist, die Energie für alle Zeiten in einem Wellenpacket eingeschlossen bleibt, dessen Fourierkoeffizienten exponentiell mit der Wellenzahl abfallen. Schliesslich zeigen wir, dass für Lösungen des FPU-Gitters mit den oben beschriebenen Anfangswerten die Energie für ein längeres Zeitintervall in einem derartigen Wellenpacket eingeschlossen bleibt, als zuvor bekannt war.

Im zweiten Teil wird die Streuabbildung für die KdV Gleichung auf $\mathbb{R}$ untersucht. Es wird gezeigt, dass für Potentiale in gewichteten Sobolev Räumen ohne Eigenzustände der nichtlineare Teil der Streuabbildung 1-regularisierend ist und die sich daraus ergebende Anwendungen für die Lösungen der KdV Gleichung diskutiert.

Abstract

In this thesis we investigate two examples of infinite dimensional integrable Hamiltonian systems in 1-space dimension: the Toda chain with periodic boundary conditions and large number of particles, and the Korteweg-de Vries (KdV) equation on $\mathbb{R}$.

In the first part of the thesis we study the Birkhoff coordinates (Cartesian action angle coordinates) of the Toda lattice with periodic boundary condition in the limit where the number N of the particles tends to infinity. We prove that the transformation introducing such coordinates maps analytically a complex ball of radius R/N^α (in discrete Sobolev-analytic norms) into a ball of radius R'/N^α (with $R, R' > 0$ independent of N) if and only if $\alpha \geq 2$. Then we consider the problem of equipartition of energy in the spirit of Fermi-Pasta-Ulam. We deduce that corresponding to initial data of size R/N^2, $0 < R \ll 1$, and with only the first Fourier mode excited, the energy remains forever in a packet of Fourier modes exponentially decreasing with the wave number. Finally we consider the original FPU model and prove that energy remains localized in a similar packet of Fourier modes for times one order of magnitude longer than those covered by previous results which is the time of formation of the packet. The proof of the theorem on Birkhoff coordinates is based on a new quantitative version of a Vey type theorem by Kuksin and Perelman which could be interesting in itself.

In the second part of the thesis we study the scattering map of the KdV on $\mathbb{R}$. We prove that in appropriate weighted Sobolev spaces of the form $H^N \cap L^2_M$, with integers $N \geq 2M \geq 8$ and in the case of no bound states, the scattering map is a perturbation of the Fourier transform by a regularizing operator. As an application of this result, we show that the difference of the KdV flow and the corresponding Airy flow is 1-smoothing.

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Contents

Introduction

In the last decades the problem of a rigorous analysis of the theory of infinite dimensional integrable Hamiltonian systems has been widely studied. In particular Kappeler with collaborators introduced a series of methods in order to construct rigorously action-angle variables for 1-dimensional integrable Hamiltonian PDE's on $\mathbb{T}$. The program succeeded in many cases, like KdV [KP03], defocusing and focusing NLS [GK14, KLTZ09]. In each case considered, it has been proved that there exists a real analytic symplectic diffeomorphism between two scales of Hilbert spaces which introduce action-angle coordinates.

The present thesis is part of this program. Infinite dimensional integrable Hamiltonian systems in 1-space dimension come up in two setups: (i) on compact intervals (finite volume) and (ii) on infinite intervals (infinite volume). The dynamical behaviour of the systems in the two setups have many similar features, but also distinct ones, mostly due to the different manifestation of dispersion. In this thesis we analyze two systems in different setups. As an example of a system in the first setup we study the Toda chain with a large number of particles, while as an example of a system in the second setup we study the Korteweg-de Vries equation (KdV) on $\mathbb{R}$. The choice of the Toda chain is motivated by the application to the FPU chain which will be discussed below, while the choice of KdV is motivated by the question if features such as the 1-smoothing property established recently for this equation in the periodic setup also hold in the infinite volume case.

We describe now in more details our results.

The Toda's chain. The Toda chain is the system with Hamiltonian

$$H_{Toda}(p, q) = \frac{1}{2} \sum_{j=0}^{N-1} p_j^2 + \sum_{j=0}^{N-1} e^{q_j - q_{j+1}} \tag{1}$$

and periodic boundary conditions $q_N = q_0$, $p_N = p_0$, which is known to be integrable. We are interested in the limit $N \to \infty$. By standard Arnold-Liouville theory the system admits action angle coordinates. However the actual introduction of such coordinates is quite complicated and the corresponding transformation has only recently been studied analytically in a series of papers by Henrici and Kappeler [HK08b, HK08c]. In particular such authors have proved the existence of global Birkhoff coordinates, namely canonical coordinates (x_k, y_k) analytic on the whole phase space, with the property that the k^{th} action is given by $(x_k^2 + y_k^2)/2$. The construction of Henrici and Kappeler, however is not uniform in the size of the chain, in the sense that the map Φ_N introducing Birkhoff coordinates is globally analytic for any fixed N, but it could (and actually does) develop singularities as $N \to +\infty$. Our main result is to prove some analyticity properties fulfilled by Φ_N uniformly in the limit $N \to +\infty$. Precisely we consider complex balls centered at the origin and prove that Φ_N maps analytically a ball of radius R/N^α in discrete Sobolev-analytic norms into a ball of radius R'/N^α (with $R, R' > 0$ independent of N) if and only if $\alpha \geq 2$. To come to a precise statement we have to introduce a suitable topology in $\mathbb{C}^{N-1} \times \mathbb{C}^{N-1}$. Consider the Toda lattice in the subspace characterized by $\sum_j q_j = 0 = \sum_j p_j$ which is invariant under the dynamics. Introduce the discrete Fourier transform $\mathcal{F}(q) = \hat{q}$ defined by

$$\hat{q}_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} q_j e^{2\mathrm{i}\pi jk/N}, \qquad k \in \mathbb{Z} \ , \tag{2}$$

and consider $\hat{p}_k$ defined analogously. Finally introduce the linear Birkhoff variables

$$X_k = \frac{\hat{p}_k + \hat{p}_{N-k} - \mathrm{i}\omega_k(\hat{q}_k - \hat{q}_{N-k})}{\sqrt{2\omega_k}} \ , \quad Y_k = \frac{\hat{p}_k - \hat{p}_{N-k} + \mathrm{i}\omega_k(\hat{q}_k + \hat{q}_{N-k})}{\mathrm{i}\sqrt{2\omega_k}} \ , \quad k = 1, ..., N-1 \ , \tag{3}$$

where $\omega_k \equiv \omega\left(\frac{k}{N}\right) := 2\sin(k\pi/N)$; using such coordinates, which are symplectic, the quadratic part of the Hamiltonian takes the form

$$H_0 = \sum_{k=1}^{N-1} \omega\left(\tfrac{k}{N}\right) \frac{X_k^2 + Y_k^2}{2} \ . \tag{4}$$

For any $s \geq 0$, $\sigma \geq 0$ introduce in $\mathbb{C}^{N-1} \times \mathbb{C}^{N-1}$ the discrete Sobolev-analytic norm

$$\|(X,Y)\|^2_{\mathcal{P}^{s,\sigma}} := \frac{1}{N} \sum_{k=1}^{N-1} [k]_N^{2s}\, e^{2\sigma [k]_N}\, \omega\left(\tfrac{k}{N}\right) \frac{|X_k|^2 + |Y_k|^2}{2} \tag{5}$$

where $[k]_N := \min(|k|, |N-k|)$. The space $\mathbb{C}^{N-1} \times \mathbb{C}^{N-1}$ endowed by such a norm will be denoted by $\mathcal{P}^{s,\sigma}$. We denote by $B^{s,\sigma}(R)$ the ball of radius R and center 0 in the topology defined by the norm $\|.\|_{\mathcal{P}^{s,\sigma}}$. We will also denote by $B^{s,\sigma}_{\mathbb{R}} := B^{s,\sigma}(R) \cap (\mathbb{R}^{N-1} \times \mathbb{R}^{N-1})$ the *real* ball of radius R. The most important result in this section is the following

Theorem 0.1. *For any $s \geq 0$, $\sigma \geq 0$ there exist strictly positive constants $R_{s,\sigma}$, $R'_{s,\sigma}$, such that for any $N \geq 2$, the map Φ_N is analytic as a map*

$$\Phi_N : \ B^{s,\sigma}\left(\frac{R_{s,\sigma}}{N^\alpha}\right) \hookrightarrow B^{s,\sigma}\left(\frac{R'_{s,\sigma}}{N^\alpha}\right) \ , \qquad (x,y) \mapsto (X,Y)$$

if and only if $\alpha \geq 2$. The same is true for the inverse mapping Φ_N^{-1}.

In order to prove the "if" part of Theorem 0.1 we apply to the Toda lattice a Vey type theorem [Vey78] for infinite dimensional systems recently proved by Kuksin and Perelman [KP10]. Actually, we need to prove a new quantitative version of Kuksin-Perelman's theorem, a result that we think could be interesting in itself.

In order to prove the "only if" part of Theorem 0.1, we explicitly construct the first term of the Taylor expansion of Φ_N through Birkhoff normal form techniques, and prove that the second differential $Q^{\Phi_N} := d^2\Phi_N(0,0)$ at the origin diverges like N^2. It follows that, as $N \to +\infty$, the real diffeomorphism Φ_N develops a singularity at zero in the second derivative. Thus, by Cauchy estimate, the image of a ball of radius R/N^α is unbounded when $\alpha < 2$.

We finally apply the result to the problem of equipartition of energy in the spirit of Fermi-Pasta-Ulam. Recall that the FPU (α, β)-model is the Hamiltonian lattice with Hamiltonian function which, in suitable rescaled variables, takes the form

$$H_{FPU}(p,q) = \sum_{j=0}^{N-1} \frac{p_j^2}{2} + U(q_j - q_{j+1}) \ , \qquad U(x) = \frac{x^2}{2} + \ \frac{x^3}{6} + \beta\frac{x^4}{24} \ . \tag{6}$$

We will consider the case of periodic boundary conditions: $q_0 = q_N$, $p_0 = p_N$. Let us denote by E_k the energy of the k^{th} normal mode, and by $\mathcal{E}_k := E_k/N$ the specific energy in the k^{th} mode. In their celebrated numerical experiment Fermi Pasta and Ulam [FPU65] studied both the behaviour of $\mathcal{E}_k(t)$ and of its time average $\langle \mathcal{E}_k \rangle(t) := \frac{1}{t}\int_0^t \mathcal{E}_k(s)ds$. They observed that, corresponding to initial data with $\mathcal{E}_1(0) \neq 0$ and $\mathcal{E}_k(0) = 0$ $\forall k \neq 1, N-1$, the quantities $\mathcal{E}_k(t)$ present a recurrent behaviour, while their averages $\langle \mathcal{E}_k \rangle(t)$ quickly relax to a sequence $\bar{\mathcal{E}}_k$ exponentially decreasing with k. This is what is known under the name of FPU packet of modes.

A systematic numerical study of the evolution of the Toda compared to FPU, paying particular attention to the dependence on N of the phenomena, was performed by Benettin and Ponno [BP11] (see also [BCP13]). In particular such authors put into evidence the fact that the FPU packet seems to have an infinite lifespan in the Toda lattice. Furthermore they showed that the relevant parameter controlling the lifespan of the packet in the FPU model is the distance of FPU from the corresponding Toda lattice, which is measured by the quantity $(\beta - 1)$.

As a corollary of Theorem 0.1 we prove that in the Toda lattice, corresponding to initial data in a ball of radius R/N^2 $(0 < R \ll 1)$ and with only the first Fourier mode excited, the energy remains forever in a packet of Fourier modes exponentially decreasing with the wave number. Then we consider the original FPU model and prove that, corresponding to the same initial data, energy remains in an exponentially localized packet of Fourier modes for very long times (see Theorem 0.3 below), namely for times one order of magnitude longer then those covered by previous results [BP06]. This is relevant in view of the fact that the time scale covered in [BP06] is that of formation of the packet, so the result that we prove allows to conclude that the packet persists over a time much longer then the one needed for its formation. It is convenient to state the results for Toda and FPU using the small parameter $\mu := \frac{1}{N}$ as in [BP06]. We prove the following theorem

Theorem 0.2. *Consider the Toda lattice* (1)*. Fix $\sigma > 0$, then there exist constants R_0, C_1, such that the following holds true. Consider an initial datum with*

$$\mathcal{E}_1(0) = \mathcal{E}_{N-1}(0) = R^2 e^{-2\sigma} \mu^4 \quad , \quad \mathcal{E}_k(0) \equiv \mathcal{E}_k(t)\big|_{t=0} = 0 \ , \quad \forall k \neq 1, N-1 \tag{7}$$

with $R < R_0$. Then, along the corresponding solution, one has

$$\mathcal{E}_k(t) \leq R^2(1 + C_1 R)\mu^4 e^{-2\sigma k} \ , \quad \forall 1 \leq k \leq \lfloor N/2 \rfloor \ , \quad \forall t \in \mathbb{R} \ . \tag{8}$$

For the FPU model we have the following theorem

Theorem 0.3. *Consider the FPU system* (6)*. Fix $s \geq 1$ and $\sigma \geq 0$; then there exist constants R_0', C_2, T, such that the following holds true. Consider a real initial datum fulfilling* (7) *with $R < R_0'$, then, along the corresponding solution, one has*

$$\mathcal{E}_k(t) \leq \frac{16 R^2 \mu^4 e^{-2\sigma k}}{k^{2s}} \ , \quad \forall 1 \leq k \leq \lfloor N/2 \rfloor \ , \quad |t| \leq \frac{T}{R^2 \mu^4} \cdot \frac{1}{|\beta - 1| + C_2 R \mu^2} \ . \tag{9}$$

Furthermore, for $1 \leq k \leq N-1$, consider the action $I_k := \frac{x_k^2 + y_k^2}{2}$ of the Toda lattice and let $I_k(t)$ be its evolution according to the FPU flow. Then one has

$$\frac{1}{N} \sum_{k=1}^{N-1} [k]_N^{2(s-1)} e^{2\sigma [k]_N} \omega\left(\tfrac{k}{N}\right) |I_k(t) - I_k(0)| \leq C_3 R^2 \mu^5 \qquad \textit{for } t \textit{ fullfilling } (9) \tag{10}$$

Let us remark that our analysis is part of a project aiming at studying the dynamics of periodic Toda lattices with a large number of particles, in particular its asymptotics. First results in this project were obtained in the papers [BKP09, BKP13b, BKP13a] (see also [BGPU03]).

They are based on the Lax pair representation of the Toda lattice in terms of periodic Jacobi matrices. The spectrum of these matrices leads to a complete set of conserved quantities and hence determines the Toda Hamiltonian and the dynamics of Toda lattices, such as their frequencies.

In order to study the asymptotics of Toda lattices for a large number N of particles one therefore needs to work in two directions: on the one hand one has to study the asymptotics of the spectrum of Jacobi matrices as $N \to \infty$ and on the other hand, one needs to use tools of the theory of integrable systems in order to effectively extract information on the dynamics of Toda lattices from the periodic spectrum of periodic Jacobi matrices.

The KdV on $\mathbb{R}$. In the second part of the thesis we show that for the KdV on the line, the scattering map is an analytic perturbation of the Fourier transform by a 1-smoothing nonlinear operator. With the application we have in mind, we choose a setup for the scattering map so that the spaces considered are left invariant under the KdV flow. Recall that the KdV equation on $\mathbb{R}$

$$\partial_t u(t,x) = -\partial_x^3 u(t,x) - 6u(t,x)\partial_x u(t,x) \ , \qquad u(0,x) = q(x) \ , \tag{11}$$

is globally in time well-posed in various function spaces such as the Sobolev spaces $H^N \equiv H^N(\mathbb{R},\mathbb{R})$, $N \in \mathbb{Z}_{\geq 2}$, as well as on the weighted spaces $H^{2N} \cap L^2_M$, with integers $N \geq M \geq 1$ [Kat66], where $L^2_M \equiv L^2_M(\mathbb{R},\mathbb{C})$ denotes the space of functions satisfying $\|q\|^2_{L^2_M} := \int_{-\infty}^{\infty}(1+|x|^2)^M |q(x)|^2 dx < \infty$.

Our analysis relies on a detailed study of the spectral data of the Schrödinger operator $L(q) := -\partial_x^2 + q$. Denote by $f_1(q,x,k)$ and $f_2(q,x,k)$ the Jost solutions, i.e. solutions of $L(q)f = k^2 f$ with asymptotics $f_1(q,x,k) \sim e^{ikx}$, $x \to \infty$, $f_2(q,x,k) \sim e^{-ikx}$, $x \to -\infty$. The eigenvalues of the operator $L(q)$ are called bound states, and a potential q will be said to be *without bound states* if $L(q)$ has no eigenvalues. Furthermore q will be said to be *generic* if the Wronskian $W(q,k) := [f_2(q,x,k), f_1(q,x,k)]$ satisfies the condition $W(q,0) \neq 0$ (see [Fad64]). We are interested in the analytic properties of the scattering map

$$S(q,k) := [f_1(q,x,k), f_2(q,x,-k)] \ .$$

which is known to linearize the KdV flow [GGKM74].

To state our result, introduce the set

$$\mathcal{Q} := \left\{ q : \mathbb{R} \to \mathbb{R} \ , \ q \in L^2_4 : q \text{ without bound states and generic} \right\} \ , \tag{12}$$

and for any integers $N \geq 0$ and $M \geq 4$ define $\mathcal{Q}^{N,M} := \mathcal{Q} \cap H^N \cap L^2_M$. We prove that for potentials $q \in \mathcal{Q}$, the scattering map $S(q,\cdot)$ takes value in the space $\mathscr{S}$ of functions $\sigma : \mathbb{R} \to \mathbb{C}$ satisfying

(S1) $\sigma(-k) = \overline{\sigma(k)}, \quad \forall k \in \mathbb{R}$;

(S2) $\sigma(0) > 0$.

Denote by $\mathscr{S}^{M,N} := \mathscr{S} \cap H^M_{\zeta,\mathbb{C}} \cap L^2_N$. Here $H^M_{\zeta,\mathbb{C}}$ is the space of functions $f \in H^{M-1}_{\mathbb{C}}$ such that the M^{th} derivative fulfills $\zeta \partial_k^M f \in L^2$, where $\zeta : \mathbb{R} \to \mathbb{R}$ is an odd monotone C^∞ function with $\zeta(k) = k$ for $|k| \leq 1/2$ and $\zeta(k) = 1$ for $k \geq 1$.

Moreover let $\mathcal{F}_\pm$ be the Fourier transformations defined by $\mathcal{F}_\pm(f) = \int_{-\infty}^{+\infty} e^{\mp 2ikx} f(x)\, dx$. In this setup, the scattering map S has the following properties:

Theorem 0.4. *For any integers $N \geq 0$, $M \geq 4$, the following holds:*

(i) The map

$$S : \mathcal{Q}^{N,M} \to \mathscr{S}^{M,N}, \quad q \mapsto S(q,\cdot)$$

is a real analytic diffeomorphism.

(ii) The maps $A := S - \mathcal{F}_-$ *and* $B := S^{-1} - \mathcal{F}_-^{-1}$ *are 1-smoothing, i.e.*

$$A : \mathcal{Q}^{N,M} \to H^M_\zeta \cap L^2_{N+1} \quad \textit{and} \quad B : \mathscr{S}^{M,N} \to H^{N+1} \cap L^2_{M-1} \ .$$

Furthermore they are real analytic maps.

Item (i) of Theorem 0.4 shows that the scattering map behaves like a nonlinear Fourier transform, interchanging the decaying and regularity properties. Item (ii) shows that the difference A of the scattering map and its linear part $\mathcal{F}_-$ is 1-smoothing.
Kappeler and Trubowitz [KT86, KT88] studied analytic properties of the scattering map S between weighted Sobolev spaces. More precisely, define the spaces

$$\begin{aligned} H^{n,\alpha} &:= \left\{ f \in L^2 : x^\beta \partial_x^j f \in L^2, 0 \le j \le n, 0 \le \beta \le \alpha \right\} \ , \\ H^{n,\alpha}_\sharp &:= \left\{ f \in H^{n,\alpha} : x^\beta \partial_x^{n+1} f \in L^2, 1 \le \beta \le \alpha \right\} \ . \end{aligned}$$

In [KT86], Kappeler and Trubowitz showed that the map $q \mapsto S(q,\cdot)$ is a real analytic diffeomorphism from $\mathcal{Q} \cap H^{N,N}$ to $\mathscr{S} \cap H^{N-1,N}_\sharp$, $N \in \mathbb{Z}_{\ge 3}$. They extend their results to potentials with finitely many bound states in [KT88]. Unfortunately, such spaces are not invariant for the KdV flow, thus they are not suited for analyzing qualitative properties of the KdV dynamic. The novelty of our work is to extend the construction of [KT86] to spaces of the form $H^N \cap L^2_M$, which, for $N \ge 2M \ge 2$, are invariant for the KdV [Kat66].

As an application of Theorem 0.4 we compare solutions of (11) to solutions of the Cauchy problem for the Airy equation on $\mathbb{R}$,

$$\partial_t v(t,x) = -\partial_x^3 v(t,x) \ , \qquad v(0,x) = p(x) \ . \tag{13}$$

Denote the flows of (13) and (11) by $U^t_{Airy}(p) := v(t,\cdot)$ respectively $U^t_{KdV}(q) := u(t,\cdot)$. We show that for $q \in \mathcal{Q}^{N,M}$ with $N \ge 2M \ge 8$, the difference $U^t_{KdV}(q) - U^t_{Airy}(q)$ is 1-smoothing, i.e. it takes values in H^{N+1}. More precisely we prove the following:

Theorem 0.5. *Let* N, M *be integers with* $N \ge 2M \ge 8$. *Then the following holds true:*

(i) $\mathcal{Q}^{N,M}$ *is invariant under the KdV flow.*

(ii) For any $q \in \mathcal{Q}^{N,M}$ *the difference* $U^t_{KdV}(q) - U^t_{Airy}(q)$ *takes values in* $H^{N+1} \cap L^2_M$. *Moreover the map*

$$\mathcal{Q}^{N,M} \times \mathbb{R}_{\ge 0} \to H^{N+1} \cap L^2_M, \qquad (q,t) \mapsto U^t_{KdV}(q) - U^t_{Airy}(q)$$

is continuous and for any fixed t *real analytic in* q.

This result is motivated from the study of the 1-smoothing property of the KdV flow in the periodic set-up, established recently in [ET13a, KST13] and addresses the question if similar results hold for the KdV flow on the line. In particular in [KST13] the 1-smoothing property of the Birkhoff map has been exploited to prove that for $q \in H^N(\mathbb{T}, \mathbb{R})$, $N \geq 1$, the difference $U^t_{KdV}(q) - U^t_{Airy}(q)$ is bounded in $H^{N+1}(\mathbb{T}, \mathbb{R})$ with a bound which grows linearly in time.

Organization of the thesis. In Chapter 1 we analyze the Toda Lattice with a large number of particles, and we prove Theorem 0.1, Theorem 0.2 and Theorem 0.3. The results of this Chapter are taken from our paper [BM14].

In Chapter 2 we analyze the KdV on $\mathbb{R}$ and we prove Theorem 0.5 and Theorem 0.4. The results of this chapter are taken from our paper [MS14].

Each chapter here is self contained and can be read separately.

Chapter 1

Birkhoff coordinates for the Toda Lattice in the limit of infinitely many particles with an application to FPU

1 Introduction and main result

It is well known that the Toda lattice, namely the system with Hamiltonian

$$H_{Toda}(p,q) = \frac{1}{2}\sum_{j=0}^{N-1} p_j^2 + \sum_{j=0}^{N-1} e^{q_j - q_{j+1}} , \tag{1.1}$$

and periodic boundary conditions $q_N = q_0$, $p_N = p_0$, is integrable [Tod67, Hén74]. Thus, by standard Arnold-Liouville theory the system admits action angle coordinates. However the actual introduction of such coordinates is quite complicated (see [FM76, FFM82]) and the corresponding transformation has only recently been studied analytically in a series of papers by Henrici and Kappeler [HK08b, HK08c]. In particular such authors have proved the existence of global Birkhoff coordinates, namely canonical coordinates (x_k, y_k) analytic on the whole $\mathbb{R}^{2N}$, with the property that the k^{th} action is given by $(x_k^2+y_k^2)/2$. The construction of Henrici and Kappeler, however is not uniform in the size of the chain, in the sense that the map Φ_N introducing Birkhoff coordinates is globally analytic for any fixed N, but it could (and actually does) develop singularities as $N \to +\infty$. Here we prove some analyticity properties fulfilled by Φ_N uniformly in the limit $N \to +\infty$. Precisely we consider complex balls centered at the origin and prove that Φ_N maps analytically a ball of radius R/N^α in discrete Sobolev-analytic norms into a ball of radius R'/N^α, with $R, R' > 0$ independent of N if and only if $\alpha \geq 2$. Furthermore we prove that the supremum of Φ_N over a complex ball of radius R/N^α diverges as $N \to +\infty$ when $\alpha < 1$.

In order to prove upper estimates on Φ_N we apply to the Toda lattice a Vey type theorem [Vey78] for infinite dimensional systems recently proved by Kuksin and Perelman [KP10]. Actually, we need to prove a new quantitative version of Kuksin-Perelman's theorem. We think that such a result could be interesting in itself.

The lower estimates on the size of Φ_N are proved by constructing explicitly the first term of the

Taylor expansion of Φ_N through Birkhoff normal form techniques; in particular we prove that the second differential $d^2\Phi_N(0)$ at the origin diverges like N^2.

We finally apply the result to the problem of equipartition of energy in the spirit of Fermi-Pasta-Ulam. We prove that in the Toda lattice, corresponding to initial data with energy E/N^3 ($0 < E \ll 1$) and with only the first Fourier mode excited, the energy remains forever in a packet of Fourier modes exponentially decreasing with the wave number. Then we consider the original FPU model and prove that, corresponding to the same initial data, energy remains in an exponentially localized packet of Fourier modes for times of order N^4 (see Theorem 1.16 below), namely for times one order of magnitude longer then those covered by previous results (see [BP06], see also [SW00, HL12]). This is relevant in view of the fact that the time scale of formation of the packet is N^3 (see [BP06]), so our result allows to conclude that the packet persists over a time much longer then the one needed for its formation.

1.1 Birkhoff coordinates for the Toda lattice

We come to a precise statement of the main results of the present chapter. Consider the Toda lattice in the subspace characterized by

$$\sum_j q_j = 0 = \sum_j p_j \tag{1.2}$$

which is invariant under the dynamics. Introduce the discrete Fourier transform $\mathcal{F}(q) = \hat{q}$ defined by

$$\hat{q}_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} q_j e^{2\mathrm{i}\pi jk/N}, \qquad k \in \mathbb{Z} , \tag{1.3}$$

and consider $\hat{p}_k$ defined analogously. Due to (1.2) one has $\hat{p}_0 = \hat{q}_0 = 0$ and furthermore $\hat{p}_k = \hat{p}_{k+N}$, $\hat{q}_k = \hat{q}_{k+N}$, $\forall k \in \mathbb{Z}$, so we restrict to $\{\hat{p}_k, \hat{q}_k\}_{k=1}^{N-1}$. Corresponding to real sequences (p_j, q_j) one has $\overline{\hat{q}_k} = \hat{q}_{N-k}$ and $\overline{\hat{p}_k} = \hat{p}_{N-k}$.

Introduce the linear Birkhoff variables

$$X_k = \frac{\hat{p}_k + \hat{p}_{N-k} - \mathrm{i}\omega_k(\hat{q}_k - \hat{q}_{N-k})}{\sqrt{2\omega_k}} \ , \quad Y_k = \frac{\hat{p}_k - \hat{p}_{N-k} + \mathrm{i}\omega_k(\hat{q}_k + \hat{q}_{N-k})}{\mathrm{i}\sqrt{2\omega_k}} \ , \quad k = 1, ..., N-1 \ , \tag{1.4}$$

where $\omega_k \equiv \omega\left(\frac{k}{N}\right) := 2\sin(k\pi/N)$; using such coordinates, which are symplectic, the quadratic part

$$H_0 := \sum_{j=0}^{N-1} \frac{p_j^2 + (q_j - q_{j+1})^2}{2} \tag{1.5}$$

of the Hamiltonian takes the form

$$H_0 = \sum_{k=1}^{N-1} \omega\left(\tfrac{k}{N}\right) \frac{X_k^2 + Y_k^2}{2} \ . \tag{1.6}$$

With an abuse of notations, we re-denote by H_{Toda} the Hamiltonian (1.1) written in the coordinates (X, Y). The following theorem is due to Henrici and Kappeler:

Theorem 1.1 ([HK08c]). *For any integer $N \geq 2$ there exists a global real analytic symplectic diffeomorphism $\Phi_N : \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \to \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$, $(X,Y) = \Phi_N(x,y)$ with the following properties:*

(i) The Hamiltonian $H_{Toda} \circ \Phi_N$ is a function of the actions $I_k := \frac{x_k^2+y_k^2}{2}$ only, i.e. (x_k, y_k) are Birkhoff variables for the Toda Lattice.

(ii) The differential of Φ_N at the origin is the identity: $d\Phi_N(0,0) = \mathbb{1}$.

Our main results concern the analyticity properties of the map Φ_N as $N \to \infty$. To come to a precise statement we have to introduce a suitable topology in $\mathbb{C}^{N-1} \times \mathbb{C}^{N-1}$.

For any $s \geq 0$, $\sigma \geq 0$ introduce in $\mathbb{C}^{N-1} \times \mathbb{C}^{N-1}$ the discrete Sobolev-analytic norm

$$\|(X,Y)\|^2_{\mathcal{P}^{s,\sigma}} := \frac{1}{N} \sum_{k=1}^{N-1} [k]_N^{2s}\, e^{2\sigma [k]_N}\, \omega\left(\tfrac{k}{N}\right)\, \frac{|X_k|^2 + |Y_k|^2}{2} \tag{1.7}$$

where

$$[k]_N := \min(|k|, |N-k|) \ .$$

The space $\mathbb{C}^{N-1} \times \mathbb{C}^{N-1}$ endowed by such a norm will be denoted by $\mathcal{P}^{s,\sigma}$. We denote by $B^{s,\sigma}(R)$ the ball of radius R and center 0 in the topology defined by the norm $\|.\|_{\mathcal{P}^{s,\sigma}}$. We will also denote by $B^{s,\sigma}_{\mathbb{R}} := B^{s,\sigma}(R) \cap (\mathbb{R}^{N-1} \times \mathbb{R}^{N-1})$ the *real* ball of radius R.

Remark 1.2. *When $\sigma = s = 0$ the norm* (1.7) *coincides with the energy norm rescaled by a factor $1/N$ (the rescaling factor will be discussed in Remark 1.11). We are particularly interested in the case $\sigma > 0$ since, in such a case, states belonging to $\mathcal{P}^{s,\sigma}$ are exponentially decreasing in Fourier space. The consideration of positive values of s will be needed in the proof of the main theorem.*

Our main result is the following Theorem.

Theorem 1.3. *For any $s \geq 0$, $\sigma \geq 0$ there exist strictly positive constants $R_{s,\sigma}$, $C_{s,\sigma}$, such that for any $N \geq 2$, the map Φ_N is analytic as a map from $B^{s,\sigma}(R_{s,\sigma}/N^2)$ to $\mathcal{P}^{s,\sigma}$ and fulfills*

$$\sup_{\|(x,y)\|_{\mathcal{P}^{s,\sigma}} \leq R/N^2} \|\Phi_N(x,y) - (x,y)\|_{\mathcal{P}^{s+1,\sigma}} \leq C_{s,\sigma} \frac{R^2}{N^2} \ , \quad \forall R < R_{s,\sigma}. \tag{1.8}$$

The same estimated is fulfilled by the inverse map Φ_N^{-1} possibly with a different $R_{s,\sigma}$.

Remark 1.4. *The estimate* (1.8) *controls the size of the nonlinear corrections in a norm which is stronger then the norm of (x,y), showing that $\Phi_N - \mathbb{1}$ is 1-smoothing. The proof of this kind of smoothing effect was actually the main aim of the work by Kuksin and Perelman [KP10], which proved it for KdV. Subsequently Kappeler, Schaad and Topalov [KST13] proved that such a smoothing property holds also globally for the KdV Birkhoff map.*

Remark 1.5. *As a consequence of* (1.8) *one has*

$$\Phi_N\left(B^{s,\sigma}\left(\frac{R}{N^2}\right)\right) \subset B^{s,\sigma}\left(\frac{R}{N^2}\left(1 + C_{s,\sigma} R\right)\right), \quad \forall R < R_{s,\sigma}, \forall N \geq 2 \tag{1.9}$$

and the same estimate is fulfilled by the inverse map Φ_N^{-1}, possibly with a different $R_{s,\sigma}$.

Corollary 1.6. *For any $s \geq 0$, $\sigma \geq 0$ there exist strictly positive constants $R_{s,\sigma}$, $C_{s,\sigma}$, with the following property. Consider the solution $v(t) \equiv (X(t), Y(t))$ of the Toda Lattice corresponding to initial data $v_0 \in B^{s,\sigma}\left(\frac{R}{N^2}\right)$ with $R \leq R_{s,\sigma}$ then one has*

$$v(t) \in B^{s,\sigma}\left(\frac{R}{N^2}(1 + C_{s,\sigma}R)\right) \ , \quad \forall t \in \mathbb{R} \ . \tag{1.10}$$

In order to state a converse of Theorem 1.3 consider the second differential $Q^{\Phi_N} := d^2\Phi_N(0,0)$ of Φ_N at the origin; $Q^{\Phi_N} : \mathcal{P}^{s,\sigma} \to \mathcal{P}^{s,\sigma}$ is a quadratic polynomial in the phase space variables[1].

Theorem 1.7. *For any $s \geq 0$, $\sigma \geq 0$ there exist strictly positive R, C, $N_{s,\sigma} \in \mathbb{N}$, such that, for any $N \geq N_{s,\sigma}$, $\alpha \in \mathbb{R}$, the quadratic form Q^{Φ_N} fulfills*

$$\sup_{v \in B^{s,\sigma}_{\mathbb{R}}\left(\frac{R}{N^\alpha}\right)} \left\| Q^{\Phi_N}(v,v) \right\|_{\mathcal{P}^{s,\sigma}} \geq CR^2 N^{2-2\alpha} \ . \tag{1.11}$$

Remark 1.8. *Roughtly speaking, one can say that, as $N \to \infty$, the real diffeomorphism Φ_N develops a singularity at zero in the second derivative.*

Using Cauchy estimate (see subsect. 3.2) one immediately gets the following corollary.

Corollary 1.9. *Assume that for some $s \geq 0$, $\sigma \geq 0$ there exist strictly positive R, R' and $\alpha \geq 0$, $\alpha' \in \mathbb{R}$, $N_{s,\sigma} \in \mathbb{N}$, s.t., for any $N \geq N_{s,\sigma}$, the map Φ_N is analytic in the complex ball $B^{s,\sigma}(R/N^\alpha)$ and fulfills*

$$\Phi_N\left(B^{s,\sigma}\left(\frac{R}{N^\alpha}\right)\right) \subset B^{s,\sigma}\left(\frac{R'}{N^{\alpha'}}\right) \ , \tag{1.12}$$

then one has $\alpha' \leq 2(\alpha - 1)$.

Remark 1.10. *A particular case of Corollary 1.9 is $\alpha < 1$, in which one has that the image of a ball of radius $RN^{-\alpha}$ under Φ_N is unbounded as $N \to \infty$.*

A further interesting case is that of $\alpha = \alpha'$, which implies $\alpha \geq 2$, thus showing that the scaling R/N^2 is the best possible one in which a property of the kind of (1.9) holds.

Remark 1.11. *A state (X, Y) is in the ball $B^{s,\sigma}(R/N^2)$ if and only if there exist interpolating periodic functions (β, α), namely functions s.t.*

$$p_j = \beta\left(\frac{j}{N}\right) \ , \quad q_j - q_{j+1} = \alpha\left(\frac{j}{N}\right) \ , \tag{1.13}$$

which are analytic in a strip of width σ and have a Sobolev-analytic norm of size R/N^2. More precisely, given a state (p, q) one considers its Fourier coefficients $(\hat{p}, \hat{q})$ and the corresponding X, Y variables; define

$$\alpha(x) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{q}_k \left(1 - e^{-2\pi \mathrm{i} k/N}\right) e^{-2\pi \mathrm{i} x k}, \qquad \beta(x) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{p}_k e^{-2\pi \mathrm{i} x k}$$

[1] actually according to the estimate (1.8) it is smooth as a map $\mathcal{P}^{s,\sigma} \to \mathcal{P}^{s+1,\sigma}$

which fulfill (1.13). *Then the Sobolev-analytic norms of* α *and* β *are controlled by* $\|(X,Y)\|_{\mathcal{P}^{s,\sigma}}$. *For example one has*

$$\|(\alpha,\beta)\|_{H^s}^2 := \|\alpha\|_{L^2}^2 + \|\beta\|_{L^2}^2 + \frac{1}{(2\pi)^{2s}}\|\partial_x^s\alpha\|_{L^2}^2 + \frac{1}{(2\pi)^{2s}}\|\partial_x^s\beta\|_{L^2}^2 = \|(X,Y)\|_{\mathcal{P}^{s,0}}^2 ,$$

where $\|\alpha\|_{L^2}^2 := \int_0^1 |\alpha(x)|^2 \, dx$. *In particular we consider here states with Sobolev-analytic norm of order* R/N^2 *with* $R \ll 1$. *The factor* $1/N$ *in the definition of the norm was introduced to get correspondence between the norm of a state and the norm of the interpolating functions.*

Remark 1.12. *As a consequence of Remark 1.11, the order in* N *of the solutions we are describing with Theorem 1.3 is the same of the solutions studied in the papers [BP06] and [BKP09, BKP13b, BKP13a].*

Remark 1.13. *The results of Theorem 1.3 and Theorem 1.7 extend to states with discrete Sobolev-Gevrey norm defined by*

$$\|(X,Y)\|_{\mathcal{P}^{s,\sigma,\nu}}^2 := \frac{1}{N}\sum_{k=1}^{N-1} [k]_N^{2s}\, e^{2\sigma [k]_N^{\nu}}\, \omega\left(\tfrac{k}{N}\right) \, \frac{|X_k|^2+|Y_k|^2}{2} \tag{1.14}$$

where $0 \leq \nu \leq 1$. *As a consequence of Remark 1.11, these states are interpolated by periodic functions with regularity Gevrey* ν.

Our analysis is part of a project aiming at studying the dynamics of periodic Toda lattices with a large number of particles, in particular its asymptotics. First results in this project were obtained in the papers [BKP09, BKP13b, BKP13a]. They are based on the Lax pair representation of the Toda lattice in terms of periodic Jacobi matrices. The spectrum of these matrices leads to a complete set of conserved quantities and hence determines the Toda Hamiltonian and the dynamics of Toda lattices, such as their frequencies. In order to study the asymptotics of Toda lattices for a large number N of particles one therefore needs to work in two directions: on the one hand one has to study the asymptotics of the spectrum of Jacobi matrices as $N \to \infty$ and on the other hand, one needs to use tools of the theory of integrable systems in order to effectively extract information on the dynamics of Toda lattices from the periodic spectrum of periodic Jacobi matrices.

The limit of a class of sequences of $N \times N$ Jacobi matrices as $N \to \infty$ has been formally studied already at the beginning of the theory of the Toda lattices (see e.g. [Tod67]). However, as pointed out in [BKP13b], these studies only allowed to (formally) compute the asymptotics of the spectrum in special cases. In particular, Toda lattices, which incorporated right and left moving waves could not be analyzed at all in this way. In [BKP13b], based on an approach pioneered in [BGPU03], the asymptotics of the spectra of sequences of Jacobi matrices corresponding to states of the form (1.13) were rigorously derived by the means of semiclassical analysis. It turns out that in such a limit the spectrum splits into three parts: one group of eigenvalues at each of the two edges of the spectrum within an interval of size $O(N^{-2})$, whose asymptotics are described by certain Hill operators, and a third group of eigenvalues, consisting of the bulk of the spectrum, whose asymptotics coincides with the one of Toda lattices at the equilibrium – see [BKP13b] for details.

In [BKP13a] the asymptotics of the eigenvalues obtained in [BKP13b] were used in order to compute the one of the actions and of the frequencies of Toda lattices. In particular it was shown that the asymptotics of the frequencies at the two edges involve the frequencies of two KdV solutions.

The tools used in [BKP13a] are those of the theory of infinite dimensional integrable systems as developed in [KP03] and adapted to the Toda lattice in [HK08b].

The present thesis takes up another important topic in the large number of particle limit of periodic Toda lattices: we study the Birkhoff coordinates near the equilibrium in the limit of large N to provide precise estimates on the size of complex balls around the equilibrium in Fourier coordinates and the corresponding size in Birkhoff coordinates. Our analysis allows to describe the evolution of Toda lattices with large number of particles in the original coordinates and to obtain an application to the study of FPU lattices (on which we will comment in the next section).

We remark that the obtained estimates on the size of the complex balls are optimal. In our view this is a strong indication that beyond such a regime the standard tools of integrable systems become inadequate for studying the asymptotic features of the dynamics of the periodic Toda lattices as $N \to \infty$.

The proofs of our results are based on a novel technique developed in [KP10] to show a Vey type theorem for the KdV equation on the circle which we adapt here to the study of Toda lattices, developing in this way another tool for the study of periodic Toda lattices with a large number of particles. We remark that for our arguments to go through, we need to assume an additional smallness condition on the set of states admitted as initial data: the states are required to be interpolated by functions α and β with Sobolev-analytic norm of size R/N^2, with $R \ll 1$ sufficiently small. (In the papers [BKP09, BKP13b, BKP13a], the size R can be arbitrarily large.)

1.2 On the FPU metastable packet

In this subsection we recall the phenomenon of the formation of a packet of modes in the FPU chain and state our related results. First of all we recall that the FPU (α, β)-model is the Hamiltonian lattice with Hamiltonian function which, in suitable rescaled variables, takes the form

$$H_{FPU}(p,q) = \sum_{j=0}^{N-1} \frac{p_j^2}{2} + U(q_j - q_{j+1}) \quad , \tag{1.15}$$

$$U(x) = \frac{x^2}{2} + \frac{x^3}{6} + \beta\frac{x^4}{24} \ . \tag{1.16}$$

We will consider the case of periodic boundary conditions: $q_0 = q_N$, $p_0 = p_N$.

Remark 1.14. *One has*

$$H_{FPU}(p,q) = H_{Toda}(p,q) + (\beta - 1)H_2(q) + H^{(3)}(q),$$

where

$$\begin{aligned} H_l(q) &:= \sum_{j=0}^{N-1} \frac{(q_j - q_{j+1})^{l+2}}{(l+2)!} \quad , \quad \forall l \geq 2 \ , \\ H^{(3)} &:= -\sum_{l \geq 3} H_l \ . \end{aligned}$$

Introduce the energies of the normal modes by

$$E_k := \frac{|\hat{p}_k|^2 + \omega\left(\frac{k}{N}\right)^2 |\hat{q}_k|^2}{2} \quad , \quad 1 \leq k \leq N-1 \ , \tag{1.17}$$

correspondingly denote by

$$\mathcal{E}_k := \frac{E_k}{N} \tag{1.18}$$

the specific energy in the k^{th} mode. Note that since p, q are real variables, one has $\mathcal{E}_k = \mathcal{E}_{N-k}$.
In their celebrated numerical experiment Fermi Pasta and Ulam [FPU65], being interested in the problem of foundation of statistical mechanics, studied both the behaviour of $\mathcal{E}_k(t)$ and of its time average

$$\langle \mathcal{E}_k \rangle(t) := \frac{1}{t} \int_0^t \mathcal{E}_k(s) ds \ .$$

They observed that, corresponding to initial data with $\mathcal{E}_1(0) \neq 0$ and $\mathcal{E}_k(0) = 0$ $\forall k \neq 1, N-1$, the quantities $\mathcal{E}_k(t)$ present a recurrent behaviour, while their averages $\langle \mathcal{E}_k \rangle(t)$ quickly relax to a sequence $\bar{\mathcal{E}}_k$ exponentially decreasing with k. This is what is known under the name of FPU packet of modes.

Subsequent numerical observations have investigated the persistence of the phenomenon for large N and have also shown that after some quite long time scale (whose precise length is not yet understood) the averages $\langle \mathcal{E}_k \rangle(t)$ relax to equipartition (see e.g. [BGG04, BGP04, BP11, BCP13]). This is the phenomenon known as metastability of the FPU packet.

The idea of exploiting the vicinity of FPU with Toda in order to study the dynamics of FPU goes back to [FFM82], in which the authors performed some numerical investigations studying the evolution of the Toda invariants in the dynamics of FPU. A systematic numerical study of the evolution of the Toda invariants in FPU, paying particular attention to the dependence on N of the phenomena, was performed by Benettin and Ponno [BP11] (see also [BCP13]). In particular such authors put into evidence the fact that the FPU packet seems to have an infinite lifespan in the Toda lattice. Furthermore they showed that the relevant parameter controlling the lifespan of the packet in the FPU model is the distance of FPU from the corresponding Toda lattice.

Our Theorem 1.3 yields as a corollary the effective existence and infinite persistence of the packet in the Toda lattice and also an estimate of its lifespan in the FPU system, estimate in which the effective parameter is the distance between Toda and FPU.

It is convenient to state the results for Toda and FPU using the small parameter

$$\mu := \frac{1}{N}$$

as in [BP06].

The following corollary is an immediate consequence of Corollary 1.6.

Corollary 1.15. *Consider the Toda lattice* (1.1)*. Fix $\sigma > 0$, then there exist constants R_0, C_1, such that the following holds true. Consider an initial datum with*

$$\mathcal{E}_1(0) = \mathcal{E}_{N-1}(0) = R^2 e^{-2\sigma} \mu^4 \quad , \quad \mathcal{E}_k(0) \equiv \mathcal{E}_k(t)\big|_{t=0} = 0 \ , \quad \forall k \neq 1, N-1 \tag{1.19}$$

with $R < R_0$. Then, along the corresponding solution, one has

$$\mathcal{E}_k(t) \leq R^2(1 + C_1 R)\mu^4 e^{-2\sigma k} \ , \quad \forall\, 1 \leq k \leq \lfloor N/2 \rfloor \ , \quad \forall t \in \mathbb{R} \ . \tag{1.20}$$

For the FPU model we have the following corollary

Theorem 1.16. *Consider the FPU system* (1.15). *Fix $s \geq 1$ and $\sigma \geq 0$; then there exist constants R_0', C_2, T, such that the following holds true. Consider a real initial datum fulfilling* (1.19) *with $R < R_0'$, then, along the corresponding solution, one has*

$$\mathcal{E}_k(t) \leq \frac{16R^2\mu^4 e^{-2\sigma k}}{k^{2s}} \ , \quad \forall\, 1 \leq k \leq \lfloor N/2 \rfloor \ , \quad |t| \leq \frac{T}{R^2\mu^4} \cdot \frac{1}{|\beta - 1| + C_2 R\mu^2} \ . \tag{1.21}$$

Furthermore, for $1 \leq k \leq N-1$, consider the action $I_k := \frac{x_k^2 + y_k^2}{2}$ of the Toda lattice and let $I_k(t)$ be its evolution according to the FPU flow. Then one has

$$\frac{1}{N} \sum_{k=1}^{N-1} [k]_N^{2(s-1)} e^{2\sigma[k]_N} \omega\left(\tfrac{k}{N}\right) |I_k(t) - I_k(0)| \leq C_3 R^2 \mu^5 \qquad \textit{for } t \textit{ fullfilling } (1.21) \tag{1.22}$$

Remark 1.17. *The estimates* (1.21) *are stronger then the corresponding estimates given in [BP06], which are*

$$\mathcal{E}_k(t) \leq C_1\mu^4 e^{-\sigma k} + C_2\mu^5 \ , \quad \forall\, 1 \leq k \leq \lfloor N/2 \rfloor \ , \quad |t| \leq \frac{T}{\mu^3}.$$

First, the time scale of validity of (1.21) *is one order longer than that of [BP06]. Second we show that as β approaches the value corresponding to the Toda lattice (1 in our units) the time of stability improves. Third the exponential estimate of $\mathcal{E}_k$ as a function of k is shown to hold also for large values of k (the μ^5 correction is missing). Finally in [BP06] it was shown that T/μ^3 is the time of formation of the metastable packet. So we can now conclude that the time of persistence of the packet is at least one order of magnitude larger (namely μ^{-4}) with respect to the time needed for its formation.*

Remark 1.18. *We recall also the result of [HL12] in which the authors obtained a control of the dynamics for longer time scales, but for initial data with much smaller energies.*

Remark 1.19. *Recently some results on energy sharing in FPU in the thermodynamic limit [MBC14](see also [Car07, CM12, GPP12]) have also been obtained, however such results are not able to explain the formation and the stability of the FPU packet of modes.*

2 A quantitative Kuksin-Perelman Theorem

2.1 Statement of the theorem

In this section we state and prove a quantitative version of Kuksin-Perelman Theorem which will be used to prove Theorem 1.3. It is convenient to formulate it in the framework of weighted ℓ^2 spaces, that we are going now to recall.

For any $N \leq \infty$, given a sequence $w = \{w_k\}_{k=1}^N$, $w_k \geq 1$ $\forall k \geq 1$, consider the space ℓ_w^2 of complex sequences $\xi = \{\xi_k\}_{k=1}^N$ with norm

$$\|\xi\|_w^2 := \sum_{k=1}^N w_k^2 |\xi_k|^2 < \infty. \tag{1.23}$$

Denote by $\mathcal{P}^w$ the complex Banach space $\mathcal{P}^w := \ell_w^2 \oplus \ell_w^2 \ni (\xi, \eta)$ endowed with the norm $\|(\xi,\eta)\|_w^2 := \|\xi\|_w^2 + \|\eta\|_w^2$. We denote by $\mathcal{P}_{\mathbb{R}}^w$ the real subspace of $\mathcal{P}^w$ defined by

$$\mathcal{P}_{\mathbb{R}}^w := \left\{ (\xi, \eta) \in \mathcal{P}^w \ : \ \eta_k = \overline{\xi}_k \ \forall\, 1 \leq k \leq N \right\}. \tag{1.24}$$

We will denote by $B^w(\rho)$ (respectively $B^w_{\mathbb{R}}(\rho)$) the ball in the topology of $\mathcal{P}^w$ (respectively $\mathcal{P}^w_{\mathbb{R}}$) with center 0 and radius $\rho > 0$.

Remark 1.20. *In the case of the Toda lattice the variables (ξ, η) are defined by*

$$\xi_k = \frac{\hat{p}_k + \mathrm{i}\omega\left(\frac{k}{N}\right)\hat{q}_k}{\sqrt{2\omega\left(\frac{k}{N}\right)}}, \quad \eta_k = \frac{\hat{p}_{N-k} - \mathrm{i}\omega\left(\frac{k}{N}\right)\hat{q}_{N-k}}{\sqrt{2\omega\left(\frac{k}{N}\right)}}, \qquad 1 \le k \le N-1 \ , \tag{1.25}$$

and their connection with the real Birkhoff variables is given by

$$X_k = \frac{\xi_k + \eta_k}{\sqrt{2}} \ , \quad Y_k = \frac{\xi_k - \eta_k}{i\sqrt{2}} \ , \quad 1 \le k \le N-1 \ . \tag{1.26}$$

We denote by $\mathcal{P}^1$ the Banach space of sequences in which all the weights w_k are equal to 1. For $\mathcal{X}, \mathcal{Y}$ Banach spaces, we shall write $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ to denote the set of linear and bounded operators from $\mathcal{X}$ to $\mathcal{Y}$. For $\mathcal{X} = \mathcal{Y}$ we will write just $\mathcal{L}(\mathcal{X})$.

Remark 1.21. *In the application to the Toda lattice with N particles we will use a finite, but not fixed N and weights of the form $w_k^2 = w_{N-k}^2 = N^3\, k^{2s}\, e^{2\sigma k}\omega\left(\frac{k}{N}\right)$, $1 \le k \le \lfloor N/2 \rfloor$.*

Given two weights w^1 and w^2, we will say that $w^1 \le w^2$ iff $w_k^1 \le w_k^2$, $\forall k$. Sometimes, when there is no risk of confusion, we will omit the index w from the different quantities.
In $\mathcal{P}^1$ we will use the scalar product

$$\left\langle (\xi^1, \eta^1), (\xi^2, \eta^2) \right\rangle_c := \sum_{k=1}^{N} \xi_k^1 \bar{\xi}_k^2 + \eta_k^1 \bar{\eta}_k^2 \ . \tag{1.27}$$

Correspondingly, the scalar product and symplectic form on the real subspace $\mathcal{P}^w_{\mathbb{R}}$ are given for $\xi^1 \equiv (\xi^1, \bar{\xi}^1)$ and $\xi^2 \equiv (\xi^2, \bar{\xi}^2)$ by

$$\left\langle \xi^1, \xi^2 \right\rangle := 2Re \sum_{k=1}^{N} \xi_k^1\, \overline{\xi_k^2} \ , \qquad \omega_0(\xi^1, \xi^2) := \left\langle E\,\xi^1, \xi^2 \right\rangle \ , \tag{1.28}$$

where $E := -\mathrm{i}$.

Given a smooth $F : \mathcal{P}^w_{\mathbb{R}} \to \mathbb{C}$, we denote by X_F the Hamiltonian vector field of F, given by $X_F = J\nabla F$, where $J = E^{-1}$. For $F, G : \mathcal{P}^w_{\mathbb{R}} \to \mathbb{C}$ we denote by $\{F, G\}$ the Poisson bracket (with respect to ω_0): $\{F, G\} := \langle \nabla F, J\nabla G \rangle$ (provided it exists). We say that the functions F, G *commute* if $\{F, G\} = 0$.

In order to state the main abstract theorem we start by recalling the notion of normally analytic map, exploited also in [Nik86] and [BG06].
First we recall that a map $\tilde{P}^r : (\mathcal{P}^w)^r \to \mathcal{B}$, with $\mathcal{B}$ a Banach space, is said to be r-*multilinear* if $\tilde{P}^r(v^{(1)}, \dots, v^{(r)})$ is linear in each variable $v^{(j)} \equiv (\xi^{(j)}, \eta^{(j)})$; a r-multilinear map is said to be *bounded* if there exists a constant $C > 0$ such that

$$\left\| \tilde{P}^r(v^{(1)}, \dots, v^{(r)}) \right\|_{\mathcal{B}} \le C \left\| v^{(1)} \right\|_w \cdots \left\| v^{(r)} \right\|_w \quad \forall v^{(1)}, \dots, v^{(r)} \in \mathcal{P}^w.$$

Correspondingly its norm is defined by

$$\left\|\tilde{P}^r\right\| := \sup_{\left\|v^{(1)}\right\|_w,\cdots,\left\|v^{(r)}\right\|_w\le 1} \left\|\tilde{P}^r(v^{(1)},\cdots,v^{(r)})\right\|_{\mathcal{B}}.$$

A map $P^r : \mathcal{P}^w \to \mathcal{B}$ is a *homogeneous polynomial* of order r if there exists a r-multilinear map $\tilde{P}^r : (\mathcal{P}^w)^r \to \mathcal{B}$ such that

$$P^r(v) = \tilde{P}^r(v,\ldots,v) \quad \forall v\in\mathcal{P}^w \ . \tag{1.29}$$

A r- homogeneous polynomial is bounded if it has finite norm

$$\|P^r\| := \sup_{\|v\|_w\le 1} \|P^r(v)\|_{\mathcal{B}} \, .$$

Remark 1.22. *Clearly $\|P^r\| \le \left\|\tilde{P}^r\right\|$. Furthermore one has $\left\|\tilde{P}^r\right\| \le e^r \|P^r\|$ – cf. [Muj86].*

It is easy to see that a multilinear map and the corresponding polynomial are continuous (and analytic) if and only if they are bounded.

Let $P^r : \mathcal{P}^w \to \mathcal{B}$ be a homogeneous polynomial of order r; assume $\mathcal{B}$ separable and let $\{\mathbf{b}_n\}_{n\ge1}\subset\mathcal{B}$ be a basis for the space $\mathcal{B}$. Expand P^r as follows

$$P^r(v) \equiv P^r(\xi,\eta) = \sum_{\substack{|K|+|L|=r\\ n\ge1}} P^{r,n}_{K,L}\xi^K\eta^L\mathbf{b}_n, \tag{1.30}$$

where $K,L\in\mathbb{N}_0^N$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $|K| := K_1+\cdots+K_N$, $\xi\equiv\{\xi_j\}_{j\ge1}$ and $\xi^K\equiv\xi_1^{K_1}\cdots\xi_N^{K_N}$, $\eta^L\equiv\eta_1^{L_1}\cdots\eta_N^{L_N}$.

Definition 1.23. *The modulus of a polynomial P^r is the polynomial $\underline{P^r}$ defined by*

$$\underline{P^r}(\xi,\eta) := \sum_{\substack{|K|+|L|=r\\ n\ge1}} \left|P^{r,n}_{K,L}\right|\xi^K\eta^L\mathbf{b}_n. \tag{1.31}$$

A polynomial P^r is said to have bounded modulus if $\underline{P^r}$ is a bounded polynomial.

A map $F:\mathcal{P}^w\to\mathcal{B}$ is said to be an *analytic germ* if there exists $\rho>0$ such that $F:B^w(\rho)\to\mathcal{B}$ is analytic. Then F can be written as a power series absolutely and uniformly convergent in $B^w(\rho)$: $F(v)=\sum_{r\ge0}F^r(v)$. Here $F^r(v)$ is a homogeneous polynomial of degree r in the variables $v=(\xi,\eta)$. We will write $F=O(v^n)$ if in the previous expansion $F^r(v)=0$ for every $r<n$.

Definition 1.24. *An analytic germ $F:\mathcal{P}^w\to\mathcal{B}$ is said to be normally analytic if there exists $\rho>0$ such that*

$$\underline{F}(v) := \sum_{r\ge0}\underline{F^r}(v) \tag{1.32}$$

is absolutely and uniformly convergent in $B^w(\rho)$. In such a case we will write $F\in\mathcal{N}_\rho(\mathcal{P}^w,\mathcal{B})$. $\mathcal{N}_\rho(\mathcal{P}^w,\mathcal{B})$ is a Banach space when endowed by the norm

$$|\underline{F}|_\rho := \sup_{v\in B^w(\rho)}\|\underline{F}(v)\|_{\mathcal{B}}. \tag{1.33}$$

Let $U\subset\mathcal{P}^w_{\mathbb{R}}$ be open. A map $F:U\to\mathcal{B}$ is said to be a real analytic germ *(respectively* real normally analytic*) on U if for each point $u\in U$ there exist a neighborhood V of u in $\mathcal{P}^w$ and an analytic germ (respectively normally analytic germ) which coincides with F on $U\cap V$.*

Remark 1.25. *It follows from Cauchy inequality that the Taylor polynomials F^r of F satisfy*

$$\|\underline{F}^r(v)\|_{\mathcal{B}} \leq |\underline{F}|_\rho \frac{\|v\|_w^r}{\rho^r} \qquad \forall v \in B^w(\rho) \ . \tag{1.34}$$

Remark 1.26. *Since $\forall r \geq 1$ one has $\|F^r\| \leq \|\underline{F}^r\|$, if $F \in \mathcal{N}_\rho(\mathcal{P}^w, \mathcal{B})$ then the Taylor series of F is uniformly convergent in $B^w(\rho)$.*

The case $\mathcal{B} = \mathcal{P}^w$ will be of particular importance; in this case the basis $\{\mathbf{b}_j\}_{j\geq 1}$ will coincide with the natural basis $\{\mathbf{e}_j\}_{j\geq 1}$ of such a space (namely the vectors with all components equal to zero except the j^{th} one which is equal to 1). We will consider also the case $\mathcal{B} = \mathcal{L}(\mathcal{P}^{w^1}, \mathcal{P}^{w^2})$ (bounded linear operators from $\mathcal{P}^{w^1}$ to $\mathcal{P}^{w^2}$), where w^1 and w^2 are weights. Here the chosen basis is $\mathbf{b}_{jk} = \mathbf{e}_j \otimes \mathbf{e}_k$ (labeled by 2 indexes).

Remark 1.27. *For $v \equiv (\xi, \eta) \in \mathcal{P}^1$, we denote by $|v|$ the vector of the modulus of the components of v: $|v| = (|v_1|, \ldots, |v_N|)$, $|v_j| := (|\xi_j|, |\eta_j|)$. If $F \in \mathcal{N}_\rho(\mathcal{P}^{w^1}, \mathcal{P}^{w^2})$ then $\underline{dF}(|v|)|u| \leq d\underline{F}(|v|)|u|$ (see [KP10]) and therefore, for any $0 < d < 1$, Cauchy estimates imply that $dF \in \mathcal{N}_{(1-d)\rho}(\mathcal{P}^{w^1}, \mathcal{L}(\mathcal{P}^{w^1}, \mathcal{P}^{w^2}))$ with*

$$|\underline{dF}|_{\rho(1-d)} \leq \frac{1}{d\rho} |\underline{F}|_\rho \ , \tag{1.35}$$

where $\underline{dF}$ is computed with respect to the basis $\mathbf{e}_j \otimes \mathbf{e}_k$.

Following Kuksin-Perelman [KP10] we will need also a further property.

Definition 1.28. *A normally analytic germ $F \in \mathcal{N}_\rho(\mathcal{P}^{w^1}, \mathcal{P}^{w^2})$ will be said to be of class $\mathcal{A}^{w^2}_{w^1,\rho}$ if $F = O(v^2)$ and the map $v \mapsto dF(v)^* \in \mathcal{N}_\rho(\mathcal{P}^{w^1}, \mathcal{L}(\mathcal{P}^{w^1}, \mathcal{P}^{w^2}))$. Here $dF(v)^*$ is the adjoint operator of $dF(v)$ with respect to the standard scalar product (1.27). On $\mathcal{A}^{w^2}_{w^1,\rho}$ we will use the norm*

$$\|F\|_{\mathcal{A}^{w^2}_{w^1,\rho}} := |\underline{F}|_\rho + \rho\, |\underline{dF}|_\rho + \rho\, |\underline{dF^*}|_\rho \, . \tag{1.36}$$

Remark 1.29. *Assume that for some $\rho > 0$ the map $F \in \mathcal{A}^{w^2}_{w^1,\rho}$, then for every $0 < d \leq \frac{1}{2}$ one has $|\underline{F}|_{d\rho} \leq 2d^2 |\underline{F}|_\rho$ and $\|F\|_{\mathcal{A}^{w^2}_{w^1,d\rho}} \leq 6d^2 \|F\|_{\mathcal{A}^{w^2}_{w^1,\rho}}$.*

A real normally analytic germ $F : B^{w^1}_{\mathbb{R}}(\rho) \to \mathcal{P}^{w^2}_{\mathbb{R}}$ will be said to be of class $\mathcal{N}_\rho(\mathcal{P}^{w^1}_{\mathbb{R}}, \mathcal{P}^{w^2}_{\mathbb{R}})$ (respectively $\mathcal{A}^{w^2}_{w^1,\rho}$) if there exists a map of class $\mathcal{N}_\rho(\mathcal{P}^{w^1}, \mathcal{P}^{w^2})$ (respectively $\mathcal{A}^{w^2}_{w^1,\rho}$), which coincides with F on $B^{w^1}_{\mathbb{R}}(\rho)$. In this case we will also denote by $|\underline{F}|_\rho$ (respectively $\|F\|_{\mathcal{A}^{w^2}_{w^1,\rho}}$) the norm defined by (1.33) (respectively (1.36)) of the complex extension of F.

Let now $F : U \subset \mathcal{P}^{w^1} \to \mathcal{P}^{w^2}$ be an analytic map. We will say that F *is real for real sequences* if $F(U \cap \mathcal{P}^{w^1}_{\mathbb{R}}) \subseteq \mathcal{P}^{w^2}_{\mathbb{R}}$, namely $F(\xi, \eta) = (F_1(\xi, \eta), F_2(\xi, \eta))$ satisfies $\overline{F_1(\xi, \bar{\xi})} = F_2(\xi, \bar{\xi})$. Clearly, the restriction $F|_{U \cap \mathcal{P}^{w^1}_{\mathbb{R}}}$ is a real analytic map.

We come now to the statement of the Vey Theorem.

Fix $\rho > 0$ and let $\Psi : B^{w^1}_{\mathbb{R}}(\rho) \to \mathcal{P}^{w^1}_{\mathbb{R}}$, $\Psi = \mathbb{1} + \Psi^0$ with $\mathbb{1}$ the identity map and $\Psi^0 \in \mathcal{A}^{w^2}_{w^1,\rho}$. Write Ψ component-wise, $\Psi = \left\{(\Psi_j, \overline{\Psi}_j)\right\}_{j\geq 1}$, and consider the foliation defined by the functions

$\left\{|\Psi_j(v)|^2/2\right\}_{j\geq 1}$. Given $v \in \mathcal{P}^w_{\mathbb{R}}$ we define the leaf through v by

$$\mathcal{F}_v := \left\{ u \in \mathcal{P}^w_{\mathbb{R}} : \frac{|\Psi_j(u)|^2}{2} = \frac{|\Psi_j(v)|^2}{2} \quad , \forall j \geq 1 \right\} \ . \tag{1.37}$$

Let $\mathcal{F} = \bigcup_{v\in\mathcal{P}^w_{\mathbb{R}}} \mathcal{F}_v$ be the collection of all the leaves of the foliation. We will denote by $T_v\mathcal{F}$ the tangent space to $\mathcal{F}_v$ at the point $v \in \mathcal{P}^w_{\mathbb{R}}$. A relevant role will also be played by the function $I = \{I_j\}_{j\geq 1}$ whose components are defined by

$$I_j(v) \equiv I_j(\xi, \bar{\xi}) := \frac{|\xi_j|^2}{2} \quad \forall j \geq 1 \ . \tag{1.38}$$

The foliation they define will be denoted by $\mathcal{F}^{(0)}$.

Remark 1.30. *Ψ maps the foliation $\mathcal{F}$ into the foliation $\mathcal{F}^{(0)}$, namely $\mathcal{F}^{(0)} = \Psi(\mathcal{F})$.*

The main theorem of this section is the following

Theorem 1.31. *(Quantitative version of Kuksin-Perelman Theorem) Let w^1 and w^2 be weights with $w^1 \leq w^2$. Consider the space $\mathcal{P}^{w^1}_{\mathbb{R}}$ endowed with the symplectic form ω_0 defined in* (1.28)*. Let $\rho > 0$ and assume $\Psi : B^{w^1}_{\mathbb{R}}(\rho) \to \mathcal{P}^{w^1}_{\mathbb{R}}$, $\Psi = \mathbb{1} + \Psi^0$ and $\Psi^0 \in \mathcal{A}^{w^2}_{w^1,\rho}$. Define*

$$\epsilon_1 := \left\|\Psi^0\right\|_{\mathcal{A}^{w^2}_{w^1,\rho}} \ . \tag{1.39}$$

Assume that the functionals $\{\frac{1}{2}|\Psi_j(v)|^2\}_{j\geq 1}$ pairwise commute with respect to the symplectic form ω_0, and that ρ is so small that

$$\epsilon_1 < 2^{-34}\rho. \tag{1.40}$$

Then there exists a real normally analytic map $\widetilde{\Psi} : B^{w^1}_{\mathbb{R}}(a\rho) \to \mathcal{P}^{w^1}_{\mathbb{R}}$, $a = 2^{-48}$, with the following properties:

i) $\widetilde{\Psi}^*\omega_0 = \omega_0$, so that the coordinates $z := \widetilde{\Psi}(v)$ are canonical;

ii) the functionals $\left\{\frac{1}{2}\left|\widetilde{\Psi}_j(v)\right|^2\right\}_{j\geq 1}$ pairwise commute with respect to the symplectic form ω_0;

iii) $\mathcal{F}^{(0)} = \widetilde{\Psi}(\mathcal{F})$, namely the foliation defined by Ψ coincides with the foliation defined by $\widetilde{\Psi}$;

iv) $\widetilde{\Psi} = \mathbb{1} + \widetilde{\Psi}^0$ with $\widetilde{\Psi}^0 \in \mathcal{A}^{w^2}_{w^1,a\rho}$ and $\left\|\widetilde{\Psi}^0\right\|_{\mathcal{A}^{w^2}_{w^1,a\rho}} \leq 2^{17}\epsilon_1$.

The following corollary holds:

Corollary 1.32. *Let $H : \mathcal{P}^{w^1}_{\mathbb{R}} \to \mathbb{R}$ be a real analytic Hamiltonian function. Let Ψ be as in Theorem 1.31 and assume that for every $j \geq 1$, $|\Psi_j(v)|^2$ is an integral of motion for H, i.e.*

$$\{H, |\Psi_j|^2\} = 0 \quad \forall\, j \geq 1. \tag{1.41}$$

Then the coordinates (x_j, y_j) defined by $x_j + iy_j = \widetilde{\Psi}_j(v)$ are real Birkhoff coordinates for H, namely canonical conjugated coordinates in which the Hamiltonian depends only on $(x_j^2 + y_j^2)/2$.

Proof of Corollary 1.32. Since $\Psi = \mathbb{1} + \Psi^0$, the functions $\Psi_j(v)$ can be used as coordinates in a suitable neighborhood of 0 in $\mathcal{P}^w_{\mathbb{R}}$. Let $\widetilde{\Psi}$ be the map in the statement of Theorem 1.31. Denote $F_l(v) := \frac{1}{2}\left|\widetilde{\Psi}_l(v)\right|^2$. Since the foliation defined by the functions $\{F_l\}_{l\geq 1}$ and the foliation defined by $\{|\Psi_j|^2\}_{j\geq 1}$ coincide (Theorem 1.31 *iii*)), each F_l is constant on the level sets of $\{|\Psi_j|^2\}_{j\geq 1}$. It follows that each F_l is a function of $\{|\Psi_j|^2\}_{j\geq 1}$ only. Since $\forall\, j \geq 1$, $|\Psi_j|^2$ is an integral of motion for H, the same is true for F_l, $\forall l \geq 1$. Define now, in a suitable neighborhood of the origin, the coordinates $(z, \bar{z})$ by $z_j \equiv \widetilde{\Psi}_j$, $\bar{z}_j \equiv \overline{\widetilde{\Psi}_j}$. Of course $F_l = \frac{|z_l|^2}{2}$. By (1.41) it follows then that

$$0 = \{H, z_l\bar{z}_l\} = \frac{1}{\mathrm{i}}\left(\frac{\partial H}{\partial z_l} z_l - \frac{\partial H}{\partial \bar{z}_l}\bar{z}_l\right). \tag{1.42}$$

Since $d\widetilde{\Psi}(0) = \mathbb{1}$ (Theorem 1.31 *iv*)), $\widetilde{\Psi}$ is invertible and its inverse $\widetilde{\Phi}$ satisfies $\widetilde{\Phi} = \mathbb{1} + \widetilde{\Phi}^0$ with $\widetilde{\Phi}^0 \in \mathcal{A}^{w^2}_{w^1,a\mu\rho}$ and $\left\|\widetilde{\Phi}^0\right\|_{\mathcal{A}^{w^2}_{w^1,a\mu\rho}} \leq 2\left\|\widetilde{\Psi}^0\right\|_{\mathcal{A}^{w^2}_{w^1,a\rho}} \leq 2^{18}\epsilon_1$ (Lemma 1.65 *ii*) in Appendix A).

Expand now $H \circ \widetilde{\Phi}$ in Taylor series in the variables $(z, \bar{z})$:

$$H \circ \widetilde{\Phi}(z, \bar{z}) = \sum_{\substack{r\geq 2,\\ |\alpha|+|\beta|=r}} H^r_{\alpha,\beta} z^\alpha \bar{z}^\beta.$$

Then equation (1.42) implies that in each term of the summation $\alpha = \beta$, therefore $H \circ \widetilde{\Phi}$ is a function of $|z_1|^2, \ldots, |z_N|^2$. Define now the real variables (x, y) as in the statement, then the claim follows immediately. □

2.2 Proof of the Quantitative Kuksin-Perelman Theorem

In this section we recall and adapt Eliasson's proof [Eli90] of the Vey Theorem following [KP10]. As we anticipated in the introduction, the novelty of our approach is to add quantitative estimates on the Birkhoff map $\widetilde{\Psi}$ of Theorem 1.31. In Appendix A we show that the class of normally analytic maps is closed under several operations like composition, inversion and flow-generation, and provide new quantitative estimates which will be used during the proof below.

The idea of the proof of Theorem 1.31 is to consider the functions $\{\Psi_j(v)\}_{j\geq 1}$ as noncanonical coordinates, and to look for a coordinate transformation introducing canonical variables and preserving the foliation $\mathcal{F}^{(0)}$ (which is the image of $\mathcal{F}$ in the noncanonical variables).

This will be done in two steps both based on the standard procedure of Darboux Theorem that we now recall. In order to construct a coordinate transformation φ transforming the closed nondegenerate form Ω_1 into a closed nondegenerate form Ω_0, then it is convenient to look for φ as the time 1 flow φ^t of a time-dependent vector field Y^t. To construct Y^t one defines $\Omega_t := \Omega_0 + t(\Omega_1 - \Omega_0)$ and imposes that

$$0 = \frac{d}{dt}\Big|_{t=0}\varphi^{t*}\Omega_t = \varphi^{t*}\left(\mathcal{L}_{Y^t}\Omega_t + \Omega_1 - \Omega_0\right) = \varphi^{t*}\left(d(Y^t \lrcorner \Omega_t) + d(\alpha_1 - \alpha_0)\right)$$

where α_1, α_0 are potential forms for Ω_1 and Ω_0 (namely $d\alpha_i = \Omega_i$, $i = 0, 1$) and $\mathcal{L}_{Y^t}$ is the Lie derivative of Y^t. Then one gets

$$Y^t \lrcorner \Omega_t + \alpha_1 - \alpha_0 = df \tag{1.43}$$

for each f smooth; then, if Ω_t is nondegenerate, this defines Y^t. If Y^t generates a flow φ^t defined up to time 1, the map $\varphi := \varphi^t|_{t=1}$ satisfies $\varphi^*\Omega_1 = \Omega_0$. Thus, given Ω_0 and Ω_1, the whole game reduces to study the analytic properties of Y^t and to prove that it generates a flow.

A non-constant symplectic form Ω will always be represented through a linear skew-symmetric invertible operator E as follows:

$$\Omega(v)(u^{(1)}; u^{(2)}) = \langle E(v)u^{(1)}; u^{(2)}\rangle \ , \quad \forall u^{(1)}, u^{(2)} \in T_v\mathcal{P}^w_{\mathbb{R}} \simeq \mathcal{P}^w_{\mathbb{R}}. \tag{1.44}$$

We denote by $\{F, G\}_\Omega$ the Poisson bracket with respect to Ω : $\{F, G\}_\Omega := \langle \nabla F, J\nabla G\rangle$, $J := E^{-1}$.

Similarly we will represent 1-forms through the vector field A such that

$$\alpha(v)(u) = \langle A(v), u\rangle, \quad \forall u \in T_v\mathcal{P}^w_{\mathbb{R}}. \tag{1.45}$$

Define $\omega_1 := (\Psi^{-1})^*\omega_0$, and let E_{ω_1} be the operator representing the symplectic form ω_1. The first step consists in transforming ω_1 to a symplectic form whose "average over $\mathcal{F}^{(0)}$" coincides with ω_0.

So we start by defining precisely what "average of k-forms" means. To this end consider the Hamiltonian vector fields $X^0_{I_l}$ of the functions $I_l \equiv \frac{|v_l|^2}{2}$ through the symplectic form ω_0; they are given by

$$X^0_{I_l}(v) = \mathrm{i}\nabla I_l(v) = \mathrm{i}v_l\mathbf{e}_l, \quad \forall\, l \geq 1. \tag{1.46}$$

For every $l \geq 1$ the corresponding flow $\phi^t_l \equiv \phi^t_{X^0_{I_l}}$ is given by

$$\phi^t_l(v) = \left(v_1, \cdots, v_{l-1}, e^{\mathrm{i}t}v_l, v_{l+1}, \cdots\right) \ .$$

Remark that the map ϕ^t_l is linear in v, 2π periodic in t and its adjoint satisfies $(\phi^t_l)^* = \phi^{-t}_l$. Given a k-form α on $\mathcal{P}^w_{\mathbb{R}}$ $(k \geq 0)$, we define its average by

$$M_j\alpha(v) = \frac{1}{2\pi}\int_0^{2\pi} ((\phi^t_j)^*\alpha)(v)dt, \quad j \geq 1 \ , \qquad \text{and} \qquad M\alpha(v) = \int_{\mathcal{T}} [(\phi^\theta)^*\alpha]\, d\theta \tag{1.47}$$

where $\mathcal{T}$ is the (possibly infinite dimensional) torus, the map $\phi^\theta = (\phi^{\theta_1}_1 \circ \phi^{\theta_2}_2 \cdots)$ and $d\theta$ is the Haar measure on $\mathcal{T}$.

Remark 1.33. *In the particular cases of 1 and 2-forms it is useful to compute the average in term of the representations* (1.44) *and* (1.45)*. Thus, for v, $u^{(1)}, u^{(2)} \in \mathcal{P}^w_{\mathbb{R}}$, if*

$$\alpha(v)u^{(1)} = \langle A(v); u^{(1)}\rangle \ , \quad \omega(v)(u^{(1)}, u^{(2)}) = \langle E(v)\, u^{(1)}; u^{(2)}\rangle \ ,$$

one has

$$(M\alpha)(v)u^{(1)} = \langle (MA)(v);\, u^{(1)}\rangle \ , \quad \textit{with} \quad MA(v) = \int_{\mathcal{T}} \phi^{-\theta}A(\phi^\theta(v))\, d\theta \tag{1.48}$$

and

$$(M\omega)(v)(u^{(1)}, u^{(2)}) = \langle (ME)(v)u^{(1)};\, u^{(2)}\rangle \ , \ \textit{with}\ ME(v) = \int_{\mathcal{T}} \phi^{-\theta}E(\phi^\theta(v))\phi^\theta\, d\theta. \tag{1.49}$$

Remark 1.34. *The operator M commutes with the differential operator d and the rotations ϕ^θ. In particular $MA(v)$ and $ME(v)$ as in* (1.48), (1.49) *satisfy*

$$\phi^\theta MA(v) = MA(\phi^\theta v), \quad \phi^\theta ME(v)u = ME(\phi^\theta v)\phi^\theta u, \qquad \forall\, \theta \in \mathcal{T} \ .$$

We study now the analytic properties of ω_1 and of its potential form α_{ω_1}. In the rest of the section denote by $S := \sum_{n=1}^{\infty} 1/n^2$ and by

$$\mu := 1/e(32S)^{1/2} \approx 0.0507 \ . \tag{1.50}$$

Lemma 1.35. *Let $\Phi := \Psi^{-1}$ and ω_1 be as above. Assume that $\epsilon_1 \leq \rho/e$. Then the following holds:*

(i) $E_{\omega_1} = -\mathrm{i} + \Upsilon_{\omega_1}$*, with* $\Upsilon_{\omega_1} \in \mathcal{N}_{\mu\rho}(\mathcal{P}^{w^1}_{\mathbb{R}}, \mathcal{L}(\mathcal{P}^{w^1}_{\mathbb{R}}, \mathcal{P}^{w^2}_{\mathbb{R}}))$ *and*

$$|\underline{\Upsilon_{\omega_1}}|_{\mu\rho} \leq \frac{8\epsilon_1}{\mu\rho} \ . \tag{1.51}$$

(ii) Define

$$W_{\omega_1}(v) := \int_0^1 \Upsilon_{\omega_1}(tv)tv \; dt \ , \tag{1.52}$$

then $W_{\omega_1} \in \mathcal{A}^{w^2}_{w^1,\mu^3\rho}$ *and* $\|W_{\omega_1}\|_{\mathcal{A}^{w^2}_{w^1,\mu^3\rho}} \leq 8\epsilon_1$*. Moreover the 1-form* $\alpha_{W_{\omega_1}} := \langle W_{\omega_1}; .\rangle$ *satisfies* $d\alpha_{W_{\omega_1}} = \omega_1 - \omega_0$.

Proof. By Lemma 1.65 one has that $\Phi = \left(\mathbb{1} + \Psi^0\right)^{-1} = \mathbb{1} + \Phi^0$ with $\Phi^0 \in \mathcal{A}^{w^2}_{w^1,\mu\rho}$ and $\left\|\Phi^0\right\|_{\mathcal{A}^{w^2}_{w^1,\mu\rho}} \leq 2\left\|\Psi^0\right\|_{\mathcal{A}^{w^2}_{w^1,\rho}} \leq 2\epsilon_1$. To prove (i), just remark that

$$E_{\omega_1}(v) = d\Phi^*(v)(-\mathrm{i})d\Phi(v) = -\mathrm{i} + d\Phi^0(v)^*(-\mathrm{i})d\Phi(v) - \mathrm{i}d\Phi^0(v) =: -\mathrm{i} + \Upsilon_{\omega_1}(v)$$

and use the results of Lemma 1.65. To prove (ii), use Poincaré construction of the potential of ω_1 which gives

$$\alpha_{\omega_1}(v)u := \langle \int_0^1 E_{\omega_1}(tv)tv, u\rangle dt = \alpha_0(v)u + \langle W_{\omega_1}(v), u\rangle, \quad W_{\omega_1}(v) = \int_0^1 \Upsilon_{\omega_1}(tv)tv \; dt \ ,$$

where α_0 is the potential for ω_0. In order to prove the analytic properties of W_{ω_1}, note that $W_{\omega_1}(v) = \int_0^1 (H_1(tv) + H_2(tv))dt$ where $H_1(v) := -\mathrm{i}\, d\Phi^0(v)v$ and $H_2(v) := d\Phi^0(v)^*(-\mathrm{i})d\Phi(v)v \equiv d\Phi^0(v)^*(-\mathrm{i}v + H_1(v))$. Thus, by Lemma 1.65, one gets that $\|H_1\|_{\mathcal{A}^{w^2}_{w^1,\mu^2\rho}} \leq 2\left\|\Phi^0\right\|_{\mathcal{A}^{w^2}_{w^1,\mu\rho}} \leq 4\epsilon_1$ and $\|H_2\|_{\mathcal{A}^{w^2}_{w^1,\mu^3\rho}} \leq 2\left\|\Phi^0\right\|_{\mathcal{A}^{w^2}_{w^1,\mu^2\rho}} \leq 4\epsilon_1$. Thus the estimate on W_{ω_1} follows. □

Remark 1.36. *One has* $M\alpha_{\omega_1} - \alpha_0 = M\alpha_{W_{\omega_1}} = \langle MW_{\omega_1}, \cdot\rangle$ *and* $\|MW_{\omega_1}\|_{\mathcal{A}^{w^2}_{w^1,\mu^3\rho}} \leq \|W_{\omega_1}\|_{\mathcal{A}^{w^2}_{w^1,\mu^3\rho}}$.

We are ready now for the first step.

Lemma 1.37. *There exists a map* $\hat{\varphi} : B^{w^1}_{\mathbb{R}}(\mu^5\rho) \to \mathcal{P}^{w^1}_{\mathbb{R}}$ *such that* $(\mathbb{1} - \hat{\varphi}) \in \mathcal{A}^{w^2}_{w^1,\mu^5\rho}$ *and*

$$\|\mathbb{1} - \hat{\varphi}\|_{\mathcal{A}^{w^2}_{w^1,\mu^5\rho}} \leq 2^5\epsilon_1 \ . \tag{1.53}$$

Moreover $\hat{\varphi}$ satisfies the following properties:

(i) $\hat{\varphi}$ *commutes with the rotations* ϕ^θ*, namely* $\phi^\theta\hat{\varphi}(v) = \hat{\varphi}(\phi^\theta v)$ *for every* $\theta \in \mathcal{T}$*.*

(ii) Denote $\hat{\omega}_1 := \hat{\varphi}^*\omega_1$*, then* $M\hat{\omega}_1 = \omega_0$*.*

Proof. We apply the Darboux procedure described at the beginning of this section with $\Omega_0 = \omega_0$ and $\Omega_1 = M\omega_1$. Then Ω_t is represented by the operator $\hat{E}^t_{\omega_1} := (-\mathrm{i} + t(ME_{\omega_1} + \mathrm{i}))$. Write equation (1.43), with $f \equiv 0$, in terms of the operators defining the symplectic forms, getting the equation $\hat{E}^t_{\omega_1}\hat{Y}^t = -MW_{\omega_1}$ (see also Remark 1.36). This equation can be solved by inverting the operator $\hat{E}^t_{\omega_1}$ by Neumann series:

$$\hat{Y}^t := -(-\mathrm{i} + tM\Upsilon_{\omega_1})^{-1}MW_{\omega_1} \ . \tag{1.54}$$

By the results of Lemma 1.35 and Remark 1.36, $\hat{Y}^t$ is of class $\mathcal{A}^{w^2}_{w^1,\mu^4\rho}$ and fulfills

$$\sup_{t\in[0,1]} \left\|\hat{Y}^t\right\|_{\mathcal{A}^{w^2}_{w^1,\mu^4\rho}} \leq 2\left\|MW_{\omega_1}\right\|_{\mathcal{A}^{w^2}_{w^1,\mu^3\rho}} \leq 2^4\epsilon_1 \ . \tag{1.55}$$

By Lemma 1.66 the vector field $\hat{Y}^t$ generates a flow $\hat{\varphi}^t : B^{w^1}_{\mathbb{R}}(\mu^5\rho) \to \mathcal{P}^{w^1}$ such that $\hat{\varphi}^t - \mathbb{1}$ is of class $\mathcal{A}^{w^2}_{w^1,\mu^5\rho}$ and satisfies

$$\left\|\hat{\varphi}^t - \mathbb{1}\right\|_{\mathcal{A}^{w^2}_{w^1,\mu^5\rho}} \leq 2\sup_{t\in[0,1]}\left\|\hat{Y}^t\right\|_{\mathcal{A}^{w^2}_{w^1,\mu^4\rho}} \leq 2^5\epsilon_1.$$

Therefore the map $\hat{\varphi} \equiv \hat{\varphi}^t|_{t=1}$ exists, satisfies the claimed estimate (1.53) and furthermore $\hat{\varphi}^*M\omega_1 = \omega_0$.

We prove now item (i). The claim follows if we show that the vector field $\hat{Y}^t$ commutes with rotations. To this aim consider equation (1.54), and define $\hat{J}^t_{\omega_1}(v) = (\hat{E}^t_{\omega_1}(v))^{-1}$. By construction the operator $\hat{E}^t_{\omega_1}$ commutes with rotations (cf. Remark 1.34), namely $\forall\,\theta_0 \in \mathcal{T}$ one has $\phi^{\theta_0}\hat{E}^t_{\omega_1}(v)u = \hat{E}^t_{\omega_1}(\phi^{\theta_0}(v))\phi^{\theta_0}u$. Then it follows that

$$\begin{aligned}\phi^{\theta_0}\hat{Y}^t(v) &= -\phi^{\theta_0}\hat{J}^t_{\omega_1}(v)MW_{\omega_1}(v) = -\hat{J}^t_{\omega_1}(\phi^{\theta_0}(v))\phi^{\theta_0}MW_{\omega_1}(v)\\ &= -\hat{J}^t_{\omega_1}(\phi^{\theta_0}(v))MW_{\omega_1}(\phi^{\theta_0}(v)) = \hat{Y}^t(\phi^{\theta_0}(v)).\end{aligned}$$

This proves item (i). Item (ii) then follows from item (i) since, defining $\hat{\omega}_1 = \hat{\varphi}^*\omega_1$, one has the chain of identities $M\hat{\omega}_1 = M\hat{\varphi}^*\omega_1 = \hat{\varphi}^*M\omega_1 = \omega_0$. □

The analytic properties of the symplectic form $\hat{\omega}_1$ can be studied in the same way as in Lemma 1.35; we get therefore the following corollary:

Corollary 1.38. *Denote by* $E_{\hat{\omega}_1}$ *the symplectic operator describing* $\hat{\omega}_1 = \hat{\varphi}^*\omega_1$*. Then*

(i) $E_{\hat{\omega}_1} = -\mathrm{i} + \Upsilon_{\hat{\omega}_1}$*, with* $\Upsilon_{\hat{\omega}_1} \in \mathcal{N}_{\mu^5\rho}(\mathcal{P}^{w^1}_{\mathbb{R}}, \mathcal{L}(\mathcal{P}^{w^1}_{\mathbb{R}}, \mathcal{P}^{w^2}_{\mathbb{R}}))$ *and* $\left|\underline{\Upsilon_{\hat{\omega}_1}}\right|_{\mu^5\rho} \leq 2^7\frac{\epsilon_1}{\mu\rho}$*.*

(ii) Define $W(v) := \int_0^1 \Upsilon_{\hat{\omega}_1}(tv)tv\,dt$*, then* $W \in \mathcal{A}^{w^2}_{w^1,\mu^7\rho}$ *and* $\|W\|_{\mathcal{A}^{w^2}_{w^1,\mu^7\rho}} \leq 2^7\epsilon_1$*.*

Furthermore the 1-form $\alpha_W := \langle W, .\rangle$ *satisfies* $d\alpha_W = \hat{\omega}_1 - \omega_0$*.*

Finally we will need also some analytic and geometric properties of the map

$$\check{\Psi} := \hat{\varphi}^{-1} \circ \Psi. \tag{1.56}$$

The functions $\{\check{\Psi}(v)\}_{j\geq 1}$ forms a new set of coordinates in a suitable neighborhood of the origin whose properties are given by the following corollary:

Corollary 1.39. *The map $\check{\Psi} : B_{\mathbb{R}}^{w^1}(\mu^8\rho) \to \mathcal{P}_{\mathbb{R}}^{w^1}$, defined in* (1.56)*, satisfies the following properties:*

(i) $d\check{\Psi}(0) = \mathbb{1}$ *and* $\check{\Psi}^0 := \check{\Psi} - \mathbb{1} \in \mathcal{A}_{w^1,\mu^8\rho}^{w^2}$ *with* $\left\|\check{\Psi}^0\right\|_{\mathcal{A}_{w^1,\mu^8\rho}^{w^2}} \leq 2^8\epsilon_1$.

(ii) $\mathcal{F}^{(0)} = \check{\Psi}(\mathcal{F})$*, namely the foliation defined by* $\check{\Psi}$ *coincides with the foliation defined by* Ψ.

(iii) The functionals $\{\frac{1}{2}\left|\check{\Psi}_j\right|^2\}_{j\geq 1}$ *pairwise commute with respect to the symplectic form* ω_0.

Proof. By Lemma 1.65 the map $\hat{\varphi}$ is invertible in $B_{\mathbb{R}}^{w^1}(\mu^6\rho)$ and $\hat{\varphi}^{-1} = \mathbb{1} + g$, with $g \in \mathcal{A}_{w^1,\mu^6\rho}^{w^2}$ and $\|g\|_{\mathcal{A}_{w^1,\mu^6\rho}^{w^2}} \leq 2^6\epsilon_1$. Then $\check{\Psi} = \mathbb{1} + \check{\Psi}^0$ where $\check{\Psi}^0 = \Psi^0 + g \circ (\mathbb{1} + \Psi^0)$. By Remark 1.29, $\left\|\Psi^0\right\|_{\mathcal{A}_{w^1,\mu^7\rho}^{w^2}} \leq 6\mu^{14}\epsilon_1$, thus Lemma 1.65 i) implies that $\check{\Psi}^0 \in \mathcal{A}_{w^1,\mu^8\rho}^{w^2}$ and moreover $\left\|\check{\Psi}^0\right\|_{\mathcal{A}_{w^1,\mu^8\rho}^{w^2}} \leq 6\mu^{14}\epsilon_1 + 2^7\epsilon_1 \leq 2^8\epsilon_1$. Item ($ii$) and ($iii$) follow from the fact that, by Lemma 1.37 (i), $\hat{\varphi}$ commutes with the rotations (see also the proof of Corollary 1.32). $\square$

The second step consists in transforming $\hat{\omega}_1$ into the symplectic form ω_0 while preserving the functions I_l. In order to perform this transformation, we apply once more the Darboux procedure with $\Omega_1 = \hat{\omega}_1$ and $\Omega_0 = \omega_0$. However, we require each leaf of the foliation to be invariant under the transformation. In practice, we look for a change of coordinates φ satisfying

$$\varphi^*\Omega_1 = \Omega_0 \ , \tag{1.57}$$

$$I_l(\varphi(v)) = I_l(v), \quad \forall\, l \geq 1 \ . \tag{1.58}$$

In order to fulfill the second equation, we take advantage of the arbitrariness of f in equation (1.43). It turns out that if f satisfies the set of differential equations given by

$$df(X_{I_l}^0) - (\alpha_1 - \alpha_0)(X_{I_l}^0) = 0, \quad \forall\, l \geq 1 \tag{1.59}$$

then equation (1.58) is satisfied (as it will be proved below). Here α_1 is the potential form of $\hat{\omega}_1$ and is given by $\alpha_1 := \alpha_0 + \alpha_W$, where α_W is defined in Corollary 1.38. However, (1.59) is essentially a system of equations for the potential of a 1-form on a torus, so there is a solvability condition. In Lemma 1.42 below we will prove that the system (1.59) has a solution if the following conditions are satisfied:

$$d(\alpha_1 - \alpha_0)|_{T\mathcal{F}^{(0)}} = 0 \ , \tag{1.60}$$

$$M(\alpha_1 - \alpha_0)|_{T\mathcal{F}^{(0)}} = 0 \ . \tag{1.61}$$

In order to show that these two conditions are fulfilled, we need a preliminary result. First, for $v \in \mathcal{P}_{\mathbb{R}}^w$ fixed, define the symplectic orthogonal of $T_v\mathcal{F}^{(0)}$ with respect to the form $\omega^t := \omega_0 + t(\hat{\omega}_1 - \omega_0)$ by

$$(T_v\mathcal{F}^{(0)})^{\angle_t} := \left\{h \in \mathcal{P}_{\mathbb{R}}^w : \omega^t(v)(u,h) = 0\ \forall u \in T_v\mathcal{F}^{(0)}\right\}. \tag{1.62}$$

Lemma 1.40. *For $v \in B^{w^1}_{\mathbb{R}}(\mu^5\rho)$, one has $T_v\mathcal{F}^{(0)} = (T_v\mathcal{F}^{(0)})^{\angle_t}$.*

Proof. First of all we have that, since for any couple of functions F, G and any change of coordinates Φ, one has

$$\{F \circ \Phi, G \circ \Phi\}_{\Phi^*\omega_0} = \{F, G\}_{\omega_0} \circ \Phi \ ,$$

it follows that

$$\{I_l, I_m\}_{\omega_1} = \left\{|\Psi_l|^2, |\Psi_m|^2\right\}_{\omega_0} = 0 \ , \quad \forall l, m \geq 1$$

and

$$\{I_l, I_m\}_{\widehat{\omega}_1} \circ \hat{\varphi}^{-1} = \left\{I_l \circ \hat{\varphi}^{-1}, I_m \circ \hat{\varphi}^{-1}\right\}_{\omega_1}$$

but, by the property of invariance with respect to rotations of $\widehat{\varphi}$ (and therefore of $\widehat{\varphi}^{-1}$), $I_j \circ \widehat{\varphi}^{-1}$ is a function of $\{I_l\}_{l\geq 1}$ only, and therefore the above quantity vanishes and one has $\forall l, m$

$$0 = \{I_l(v), I_m(v)\}_{\hat{\omega}_1} = \langle \nabla I_l(v), J_{\hat{\omega}_1}(v)\nabla I_m(v)\rangle = \langle v_l\mathbf{e}_l, J_{\hat{\omega}_1}(v)v_m\mathbf{e}_m\rangle \quad \forall\, l, m \geq 1. \tag{1.63}$$

Define $\Sigma_v := \operatorname{span}\{v_l\mathbf{e}_l,\ l \geq 1\}$. The identities (1.63) imply that $J_{\hat{\omega}_1}(v)(\Sigma_v) \subseteq \Sigma_v^{\perp} \equiv \mathrm{i}\Sigma_v$. By Corollary 1.38 (i), $E_{\hat{\omega}_1}(v)$ is an isomorphism for $v \in B^{w^1}_{\mathbb{R}}(\mu^5\rho)$, so the same is true for its inverse $J_{\hat{\omega}_1}(v)$. Hence $J_{\hat{\omega}_1}(v)(\Sigma_v) = \mathrm{i}\Sigma_v$ and $\Sigma_v = E_{\hat{\omega}_1}(v)(\mathrm{i}\Sigma_v)$ and

$$\hat{\omega}_1(X^0_{I_l}, X^0_{I_m}) = \langle E_{\hat{\omega}_1}(v)(\mathrm{i}v_l\mathbf{e}_l), \mathrm{i}v_m\mathbf{e}_m\rangle = 0, \quad \forall\, l, m \geq 1. \tag{1.64}$$

Since ω^t is a linear combination of ω_0 and $\hat{\omega}_1$, the previous formula implies that $\omega^t(v)(X^0_{I_l}, X^0_{I_m}) = 0$ for every $t \in [0, 1]$ and $v \in B^{w^1}_{\mathbb{R}}(\mu^5\rho)$, hence $T_v\mathcal{F}^{(0)} \subseteq (T_v\mathcal{F}^{(0)})^{\angle_t}$. Now assume by contradiction that the inclusion is strict: then there exists $u \in (T_v\mathcal{F}^{(0)})^{\angle_t}$, $\|u\| = 1$, such that $u \notin T_v\mathcal{F}^{(0)}$. Decompose $u = u_\top + u_\perp$ with $u_\top \in T_v\mathcal{F}^{(0)}$ and $u_\perp \in (T_v\mathcal{F}^{(0)})^\perp$. Due to the bilinearity of $\omega(v)^t$, we can always assume that $u \equiv u_\perp$. Then for every $l \geq 1$

$$dI_l(v)(-\mathrm{i}u) = \langle \nabla I_l(v), -\mathrm{i}u\rangle = \left\langle -\mathrm{i}X^0_{I_l}(v), -\mathrm{i}u\right\rangle = \left\langle X^0_{I_l}(v), u\right\rangle = 0 \qquad \forall\, l \geq 1$$

since $X^0_{I_l}(v) \in T_v\mathcal{F}^{(0)}$. Hence $\mathrm{i}u \in T_v\mathcal{F}^{(0)}$ and therefore $\omega^t(v)(-\mathrm{i}u, u) = 0$. Furthermore it holds that

$$\omega^t(0)(\mathrm{i}u, u) = \omega_0(-\mathrm{i}u, u) = \left\langle \mathrm{i}^2u, u\right\rangle = -1.$$

It follows that for $v \in B^{w^1}_{\mathbb{R}}(\mu^5\rho)$ one has $\|tM\Upsilon_{\hat{\omega}_1}(v)\|_{\mathcal{L}(\mathcal{P}^{w^1}_{\mathbb{R}}, \mathcal{P}^{w^1}_{\mathbb{R}})} \leq 1/2$, thus $\omega^t(v)(\mathrm{i}u, u) = -1 + \langle tM\Upsilon_{\hat{\omega}_1}(v)\mathrm{i}u, u\rangle < 0$, leading to a contradiction. $\square$

We can now prove the following lemma:

Lemma 1.41. *The solvability conditions* (1.60), (1.61) *are fulfilled.*

Proof. Condition (1.60) follows by equation (1.64), since

$$d(\alpha_1 - \alpha_0)(X^0_{I_l}, X^0_{I_m}) = \hat{\omega}_1(X^0_{I_l}, X^0_{I_m}) - \omega_0(X^0_{I_l}, X^0_{I_m}) = 0, \qquad \forall l, m \geq 1.$$

We analyze now (1.61). We claim that in order to fulfill this condition, one must have that $\hat{\omega}_1$ satisfies $M\hat{\omega}_1 = \omega_0$, which holds by Lemma 1.37 (ii). Indeed, since

$$0 = M\hat{\omega}_1 - \omega_0 = M(\hat{\omega}_1 - \omega_0) = Md(\alpha_1 - \alpha_0) = dM(\alpha_1 - \alpha_0),$$

there exists a function g such that $M(\alpha_1 - \alpha_0) = dg$. But $Mdg = M(M(\alpha_1 - \alpha_0)) = M(\alpha_1 - \alpha_0) = dg$, therefore $g = Mg$, so g is invariant by rotations. Hence $0 = \frac{d}{dt}\big|_{t=0} g(\phi^t_l) = dg(X^0_{I_l}) = M(\alpha_1 - \alpha_0)(X^0_{I_l})$, $\forall l \geq 1$, thus also (1.61) is satisfied. $\square$

We show now that the system (1.59) can be solved and its solution has good analytic properties:

Lemma 1.42. *(Moser) If conditions* (1.60) *and* (1.61) *are fulfilled, then equation* (1.59) *has a solution f. Moreover, denoting $h_j := (\alpha_1 - \alpha_0)(X^0_{I_j})$, the solution f is given by the explicit formula*

$$f(v) = \sum_{j=1}^{\infty} f_j(v), \qquad f_j(v) = M_1 \cdots M_{j-1} L_j h_j \tag{1.65}$$

where

$$L_j g = \frac{1}{2\pi} \int_0^{2\pi} t g(\phi_j^t) dt \ .$$

Finally $f \in \mathcal{N}_{\mu^7\rho}(\mathcal{P}^{w^1}_{\mathbb{R}}, \mathbb{C})$, $\nabla f \in \mathcal{N}_{\mu^7\rho}(\mathcal{P}^{w^1}_{\mathbb{R}}, \mathcal{P}^{w^2}_{\mathbb{R}})$ and

$$\left|\underline{f}\right|_{\mu^7\rho} \le 2^{10}\epsilon_1\mu^7\rho, \quad \left|\underline{\nabla f}\right|_{\mu^7\rho} \le 2^{11}\epsilon_1 \ . \tag{1.66}$$

Proof. Denote by θ_j the time along the flow generated by $X^0_{I_j}$, then one has $dg(X^0_{I_j}) = \frac{\partial g}{\partial \theta_j}$, so that the equations to be solved take the form

$$\frac{\partial f}{\partial \theta_j} = h_j, \qquad \forall j \ge 1. \tag{1.67}$$

Clearly $\frac{\partial}{\partial \theta_j} M_j h_j = 0$, and by (1.60) it follows that

$$\frac{\partial}{\partial \theta_l} M_j h_j = M_j \frac{\partial h_j}{\partial \theta_l} = M_j \frac{\partial h_l}{\partial \theta_j} = \frac{\partial}{\partial \theta_j} M_j h_l = 0, \qquad \forall l, j \ge 1,$$

which shows that $M_j h_j$ is independent of all the θ's, thus $M_j h_j = M h_j$. Furthermore, by (1.61) one has $M h_j = 0$, $\forall\, j \ge 1$. Now, using that $\frac{\partial}{\partial \theta_j} L_j g = g - M_j g$, one verifies that f_j defined in (1.65) satisfies

$$\frac{\partial f_j}{\partial \theta_l} = \begin{cases} 0 & \text{if } l < j \\ M_1 \cdots M_{j-1} h_j & \text{if } l = j \\ M_1 \cdots M_{j-1} h_l - M_1 \cdots M_j h_l & \text{if } l > j \end{cases}$$

where, for $j = 1$, we defined $M_1 \cdots M_{j-1} h_l = h_l$. Thus the series $f(v) := \sum_{j \ge 1} f_j(v)$, if convergent, satisfies (1.67).
We prove now the convergence of the series for f and ∇f. First we define, for $\theta \in \mathcal{T}$,

$$\Theta_j^\theta := \phi_1^{\theta_1} \cdots \phi_j^{\theta_j} \qquad \forall\, j \ge 1 \ ,$$

then by (1.65) one has

$$f_j(v) = \int_{\mathcal{T}^j} \theta_j h_j(\Theta_j^\theta v) \, d\theta^j \ , \tag{1.68}$$

$$\nabla f_j(v) = \int_{\mathcal{T}^j} \Theta_j^{-\theta} \theta_j \nabla h_j(\Theta_j^\theta v) \, d\theta^j \ , \tag{1.69}$$

where $\mathcal{T}^j$ is the j-dimensional torus and $d\theta^j = \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_j}{2\pi}$. Now, using that

$$h_j(v) = \langle W(v), X^0_{I_j}(v) \rangle = Re(\mathrm{i} W_j(v) \bar{v}_j) \qquad \forall\, j \ge 1$$

one gets that $\underline{f_j}(|v|) \leq 2\pi\, \underline{h_j}(|v|) \leq 2\pi\, \underline{W_j}(|v|)|v_j|$, therefore $\underline{f}(|v|) \leq \sum_{j=1}^{\infty} \underline{f_j}(|v|) \leq 2\pi\, \left\|\underline{W}(|v|)\right\|_{w^1} \left\|v\right\|_{w^1}$ and it follows that $\left|\underline{f}\right|_{\mu^7\rho} \leq 2\pi\, \left|\underline{W}\right|_{\mu^7\rho} \mu^7\rho$. This proves the convergence of the series defining f. Consider now the gradient of h_j, whose k^{th} component is given by

$$
[\nabla h_j(v)]_k = Re\left(\mathrm{i}\frac{\partial W_j(v)}{\partial v_k}\bar{v}_j\right) + \delta_{j,k}\, Re\left(\mathrm{i}W_j(v)\right)\ .
$$

Inserting the formula displayed above in (1.69) we get that ∇f_j is the sum of two terms. We begin by estimating the second one, which we denote by $(\nabla f_j)^{(2)}$. The k^{th} component of $(\nabla f)^{(2)} := \sum_j (\nabla f_j)^{(2)}$ is given by

$$
\left[(\nabla f(v))^{(2)}\right]_k = \left[\sum_j (\nabla f_j(v))^{(2)}\right]_k = \int_{\mathcal{T}^k} \Theta_k^{-\theta}\theta_k\, Re\left(\mathrm{i}W_k(\Theta_k^{\theta}v)\right)\, d\theta^k\ , \tag{1.70}
$$

thus, for any $v \in B_{\mathbb{R}}^{w^1}(\mu^7\rho)$ one has $\left[(\underline{\nabla f}(|v|))^{(2)}\right]_k \leq 2\pi\, \underline{W_k}(|v|)$, and therefore

$$
\left|\underline{(\nabla f)^{(2)}}\right|_{\mu^7\rho} \leq 2\pi\, \left|\underline{W}\right|_{\mu^7\rho} \leq \pi 2^8\epsilon_1.
$$

We come to the other term, which we denote by $(\nabla f_j)^{(1)}$. Its k^{th} component is given by

$$
\left[(\nabla f_j(v))^{(1)}\right]_k = \int_{\mathcal{T}^j} \Theta_j^{-\theta}\theta_j Re\left(\mathrm{i}\frac{\partial W_j}{\partial v_k}(\Theta_j^{\theta}v)\overline{\phi_j^{\theta_j}v_j}\right) d\theta\ . \tag{1.71}
$$

Then $\underline{\nabla f_j}(|v|) \leq 2\pi\underline{\frac{\partial W_j}{\partial v_k}}(|v|)|v_j| = 2\pi[\underline{dW}(|v|)]_k^j|v_j|$.
It follows that the k^{th} component of the function $(\nabla f)^{(1)} := \sum_j (\nabla f_j)^{(1)}$ satisfies

$$
\left[(\underline{\nabla f}(|v|))^{(1)}\right]_k \leq \left[\sum_j (\underline{\nabla f_j}(|v|))^{(1)}\right]_k \leq 2\pi\, \sum_j [\underline{dW}(|v|)]_k^j|v_j|\ .
$$

Therefore $\left|\underline{(\nabla f)^{(1)}}\right|_{\mu^7\rho} \leq 2\pi\, \left\|W\right\|_{\mathcal{A}^{w^2}_{w^1,\mu^7\rho}} \leq \pi 2^8\epsilon_1$. This is the step at which the control of the norm of the modulus $\underline{dW^*}$ of dW^* is needed. Thus the claimed estimate for ∇f follows. □

We can finally apply the Darboux procedure in order to construct an analytic change of coordinates φ which satisfies (1.57) and (1.58).

Lemma 1.43. *There exists a map $\varphi : B_{\mathbb{R}}^{w^1}(\mu^9\rho) \to \mathcal{P}_{\mathbb{R}}^{w^1}$ which satisfies* (1.57). *Moreover $\varphi - \mathbb{1} \in \mathcal{N}_{\mu^9\rho}(\mathcal{P}_{\mathbb{R}}^{w^1}, \mathcal{P}_{\mathbb{R}}^{w^2})$, $\varphi - \mathbb{1} = O(v^2)$ and*

$$
\left|\underline{\varphi - \mathbb{1}}\right|_{\mu^9\rho} \leq 2^{14}\epsilon_1\ . \tag{1.72}
$$

Proof. As anticipated just after Corollary 1.39, we apply the Darboux procedure with $\Omega_0 = \omega_0$, $\Omega_1 = \hat{\omega}_1$ and f solution of (1.59). Then equation (1.43) takes the form

$$
Y^t = (-\mathrm{i} + t\Upsilon_{\hat{\omega}_1})^{-1}(\nabla f - W), \tag{1.73}
$$

where $\Upsilon_{\hat{\omega}_1}$ and W are defined in Corollary 1.38. By Lemma 1.42 and Corollary 1.38, the vector field Y^t is of class $\mathcal{N}_{\mu^8\rho}(\mathcal{P}_{\mathbb{R}}^{w^1}, \mathcal{P}_{\mathbb{R}}^{w^2})$ and

$$\sup_{t\in[0,1]} \left|\underline{Y^t}\right|_{\mu^8\rho} < 2(2^{11}\epsilon_1 + 2^7\epsilon_1) < 2^{13}\epsilon_1.$$

Thus Y^t generates a flow $\varphi^t : B_{\mathbb{R}}^{w^1}(\mu^9\rho) \to \mathcal{P}_{\mathbb{R}}^{w^1}$, defined for every $t \in [0,1]$, which satisfies (cf. Lemma 1.66)

$$\left|\underline{\varphi^t - \mathbb{1}}\right|_{\mu^9\rho} \leq 2^{14}\epsilon_1, \quad \forall t \in [0,1]\ .$$

Thus the map $\varphi := \varphi^t|_{t=1}$ exists and satisfies the claimed properties. □

We prove now that the map φ of Lemma 1.43 satisfies also equation (1.58).

Lemma 1.44. *Let f be as in* (1.65) *and φ^t be the flow map of the vector field Y^t defined in* (1.73)*. Then $\forall\, l \geq 1$ one has $I_l(\varphi^t(v)) = I_l(v)$, for each $t \in [0,1]$.*

Proof. The following chain of equivalences follows from Lemma 1.40 and the Darboux equation (1.43):

$$\begin{aligned} I_l(\varphi^t(v)) = I_l(v) &\iff 0 = \frac{d}{dt} I_l(\varphi^t(v)) = dI_l(Y^t(v)) \iff Y^t(v) \in T_v\mathcal{F}^{(0)} \\ &\iff Y^t(v) \in (T_v\mathcal{F}^{(0)})^{\angle_t} \iff \left(\omega_v^t(Y^t(v), X_{I_l}^0(v)) = 0\ ,\ \forall l \geq 1\right) \\ &\iff \alpha_1(X_{I_l}^0) - \alpha_0(X_{I_l}^0) = df(X_{I_l}^0) \quad \forall l \geq 1\ . \end{aligned}$$

In turn the last property follows since f is a solution of (1.59). □

We can finally prove the quantitative version of the Kuksin-Perelman Theorem.

Proof of Theorem 1.31. Consider the map φ of Lemma 1.43. Since $d\varphi(0) = \mathbb{1}$, φ is invertible in $B_{\mathbb{R}}^{w^1}(\mu^{10}\rho)$ and $\varphi^{-1} = \mathbb{1} + g_1$ with $g_1 \in \mathcal{N}_{\mu^{10}\rho}(\mathcal{P}_{\mathbb{R}}^{w^1}, \mathcal{P}_{\mathbb{R}}^{w^2})$ and $\left|\underline{g_1}\right|_{\mu^{10}\rho} \leq 2\left|\underline{\varphi - \mathbb{1}}\right|_{\mu^9\rho} \leq 2^{15}\epsilon_1$ (cf. Lemma 1.64). Define now

$$\widetilde{\Psi} := \varphi^{-1} \circ \check{\Psi}.$$

It's easy to check that $\widetilde{\Psi}^*\omega_0 = \omega_0$, thus proving that $\widetilde{\Psi}$ is symplectic. By equation (1.58) one has $I_l(\widetilde{\Psi}(v)) = I_l(\check{\Psi}(v))$ for every $l \geq 1$, therefore $\widetilde{\Psi}$ and $\check{\Psi}$ define the same foliation, which coincides also with the foliation defined by Ψ, c.f. Corollary 1.39. Similarly one proves that the functionals $\left\{\frac{1}{2}\left|\widetilde{\Psi_j}(v)\right|\right\}_{j\geq 1}$ pairwise commute with respect to the symplectic form ω_0. We have thus proved item $i) - iii)$ of Theorem 1.31.

We prove now item $iv)$. Clearly $d\widetilde{\Psi}(0) = \mathbb{1}$, and $\widetilde{\Psi}^0 := \widetilde{\Psi} - \mathbb{1} = \check{\Psi}^0 + g_1 \circ (\mathbb{1} + \check{\Psi}^0)$ is of class $\mathcal{N}_{\mu^{11}\rho}(\mathcal{P}_{\mathbb{R}}^{w^1}, \mathcal{P}_{\mathbb{R}}^{w^2})$. Moreover, by Remark 1.29 and Corollary 1.39 (i), one has $\left|\underline{\check{\Psi}^0}\right|_{\mu^{11}\rho} \leq 2\mu^6\left|\underline{\check{\Psi}^0}\right|_{\mu^8\rho} \leq \mu^6 2^9\epsilon_1 \leq \mu^{11}\rho$ by condition (1.40). Thus $\left|\underline{\mathbb{1} + \check{\Psi}^0}\right|_{\mu^{11}\rho} \leq \mu^{10}\rho$ and by Lemma 1.63

$$\left|\underline{\widetilde{\Psi}_0}\right|_{\mu^{11}\rho} \leq \left|\underline{\check{\Psi}^0}\right|_{\mu^{11}\rho} + \left|\underline{g_1 \circ (\mathbb{1} + \check{\Psi}^0)}\right|_{\mu^{11}\rho} \leq \left|\underline{\check{\Psi}^0}\right|_{\mu^{11}\rho} + \left|\underline{g_1}\right|_{\mu^{10}\rho} \leq 2^8\epsilon_1 + 2^{15}\epsilon_1 \leq 2^{16}\epsilon_1.$$

We are left to prove that $\widetilde{\Psi}^0 \in \mathcal{A}^{w^2}_{w^1,\mu^{12}\rho}$. Since $\widetilde{\Psi}^*\omega_0 = \omega_0$, one has $d\widetilde{\Psi}(v)^*(-\mathrm{i})\,\widetilde{\Psi}(v) = -\mathrm{i}$, from which it follows that $\widetilde{\Psi}^0$ satisfies

$$d\widetilde{\Psi}^0(v)^* = \mathrm{i}\, d\widetilde{\Psi}^0(v) \left(\mathbb{1} + d\widetilde{\Psi}^0(v)\right)^{-1} \mathrm{i}$$

and therefore $\widetilde{\Psi}^0 \in \mathcal{A}^{w^2}_{w^1,\mu^{12}\rho}$ with $\left\|\widetilde{\Psi}^0\right\|_{\mathcal{A}^{w^2}_{w^1,\mu^{12}\rho}} < 2^{17}\epsilon_1$. □

3 Toda lattice

3.1 Proof of Theorem 1.3 and Corollary 1.6.

We consider the Toda lattice with N particles and periodic boundary conditions on the positions q and momenta p: $q_{j+N} = q_j$, $p_{j+N} = p_j$, $\forall\, j \in \mathbb{Z}$. As anticipated in Section 1, we restrict to the invariant subspace characterized by (1.2). The phase space of the system is $\mathcal{P}^{s,\sigma}$, where $s \geq 0$, $\sigma \geq 0$ and it is defined in terms of the linear, complex, Birkhoff variables (ξ, η) (defined in (1.25)). We endow the phase space with the symplectic form [2] $\Omega_0 = -\mathrm{i} \sum_{k=1}^{N-1} d\xi_k \wedge d\eta_k$.

We will denote by $\mathcal{P}^{s,\sigma}_{\mathbb{R}}$ the real subspace of $\mathcal{P}^{s,\sigma}$ in which $\eta_k = \overline{\xi_k}$ $\forall 1 \leq k \leq N-1$, endowed with the norm (1.7), and by $B^{s,\sigma}_{\mathbb{R}}(\rho)$ the ball in $\mathcal{P}^{s,\sigma}_{\mathbb{R}}$ with center 0 and radius $\rho > 0$. The main step of the proof of Theorem 1.3 is the construction of the functions $\{\Psi_j\}_{1\leq j\leq N-1}$. This is based on a detailed analysis of the spectrum of the Jacobi matrix appearing in the Lax pair representation of the Toda lattice. So we start by recalling the elements of the theory needed for our development. Introduce the translated Flaschka coordinates [Fla74] by

$$(b, a) = \Theta(p, q), \qquad (b_j,\ a_j) := (-p_j,\ e^{\frac{1}{2}(q_j - q_{j+1})} - 1). \tag{1.75}$$

The translation of the a variables by 1 is useful in order to keep the equilibrium point at $(b, a) = (0, 0)$. Recall that the variables b, a are constrained by the conditions

$$\sum_{j=0}^{N-1} b_j = 0,\ \prod_{j=0}^{N-1} (1 + a_j) = 1\ .$$

Introduce Fourier variables $(\hat{b}, \hat{a})$ for the Flaschka coordinates by (1.3). In these variables

$$E_k = \frac{|\hat{b}_k|^2 + 4|\hat{a}_k|^2}{2} + O(\hat{a}^3), \qquad 1 \leq k \leq N-1\ . \tag{1.76}$$

The Jacobi matrix whose spectrum forms a complete set of integrals of motions for the Toda lattice

[2] so that the Hamilton equations become

$$\dot{\xi}_k = \mathrm{i}\frac{\partial H}{\partial \eta_k}, \qquad \dot{\eta}_k = -\mathrm{i}\frac{\partial H}{\partial \xi_k}\ , \tag{1.74}$$

is given by [vM76]

$$L(b,a) := \begin{pmatrix} b_0 & 1+a_0 & 0 & \dots & 1+a_{N-1} \\ 1+a_0 & b_1 & 1+a_1 & \ddots & \vdots \\ 0 & 1+a_1 & b_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1+a_{N-2} \\ 1+a_{N-1} & \dots & 0 & 1+a_{N-2} & b_{N-1} \end{pmatrix}. \tag{1.77}$$

It is useful to double the size of $L(b,a)$, redefining

$$L_{b,a} := \left(\begin{array}{cccc|cccc} b_0 & 1+a_0 & \dots & 0 & 0 & \dots & 0 & 1+a_{N-1} \\ 1+a_0 & b_1 & \ddots & \vdots & 0 & \dots & & 0 \\ \vdots & \ddots & \ddots & 1+a_{N-2} & \vdots & & & \vdots \\ 0 & \ddots & 1+a_{N-2} & b_{N-1} & 1+a_{N-1} & \dots & 0 & 0 \\ \hline 0 & \dots & 0 & 1+a_{N-1} & b_0 & 1+a_0 & \dots & 0 \\ 0 & \dots & & 0 & 1+a_0 & b_1 & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & 1+a_{N-2} \\ 1+a_{N-1} & \dots & 0 & 0 & 0 & \ddots & 1+a_{N-2} & b_{N-1} \end{array}\right). \tag{1.78}$$

Consider the eigenvalues of $L_{b,a}$ and order them in the non-decreasing sequence

$$\lambda_0(b,a) < \lambda_1(b,a) \leq \lambda_2(b,a) < \ldots < \lambda_{2N-3}(b,a) \leq \lambda_{2N-2}(b,a) < \lambda_{2N-1}(b,a)$$

where one has that where the sign $\leq$ appears equality is possible, while it is impossible in the correspondence of a sign $<$. Define the quantities

$$\gamma_j(b,a) := \lambda_{2j}(b,a) - \lambda_{2j-1}(b,a), \qquad 1 \leq j \leq N-1; \tag{1.79}$$

$\gamma_j(b,a)$ is called j^{th} *spectral gap*. The quantities $\{\gamma_j^2\}_{1\leq j\leq N-1}$ form a complete set of commuting integrals of motions, which are regular also at $(b,a) = (0,0)$. Furthermore one has $H(b,a) = H(\gamma_1^2(b,a), \ldots, \gamma_{N-1}^2(b,a))$ [BGGK93]. A spectral gap is said to be *closed* if $\gamma_j(b,a) = 0$.

The following Theorem 1.45 ensures that the assumptions of Theorem 1.31 are fulfilled by the Toda lattice.

Theorem 1.45. *There exists $\epsilon_* > 0$, independent of N, and an analytic map*

$$\Psi : \left(B^{s,\sigma}\left(\frac{\epsilon_*}{N^2}\right), \Omega_0\right) \to \mathcal{P}^{s,\sigma}, \quad (\xi,\eta) \mapsto (\phi(\xi,\eta), \psi(\xi,\eta)) \tag{1.80}$$

such that:

(Ψ1) *Ψ is real for real sequences, namely $\overline{\phi_k(\xi,\bar{\xi})} = \psi_k(\xi,\bar{\xi})$ $\forall k$.*

(Ψ2) *For every $1 \leq j \leq N-1$, and for $(\phi,\psi) \in B^{s,\sigma}\left(\frac{\epsilon_*}{N^2}\right) \cap \mathcal{P}^{s,\sigma}_{\mathbb{R}}$, one has*

$$\gamma_j^2 = \tfrac{2}{N}\omega\left(\tfrac{j}{N}\right)|\psi_j|^2 = \tfrac{2}{N}\omega\left(\tfrac{j}{N}\right)|\varphi_j|^2 \ .$$

(Ψ3) $\Psi(0,0) = (0,0)$ *and* $d\Psi(0,0) = \mathbb{1}$.

(Ψ4) *There exist constants* $C_1, C_2 > 0$*, independent of* N*, such that for every* $0 < \epsilon \leq \epsilon_*$*, the map* $\Psi^0 := \Psi - \mathbb{1} \in \mathcal{N}_{\epsilon/N^2}\left(\mathcal{P}^{s,\sigma}, \mathcal{P}^{s+1,\sigma}\right)$ *and* $[d\Psi^0]^* \in \mathcal{N}_{\epsilon/N^2}\left(\mathcal{P}^{s,\sigma}, \mathcal{L}(\mathcal{P}^{s,\sigma}, \mathcal{P}^{s+1,\sigma})\right)$*. Furthermore one has*

$$\left|\underline{\Psi^0}\right|_{\epsilon/N^2} \leq C_1 \frac{\epsilon^2}{N^2}; \qquad \left|\underline{[d\Psi^0]^*}\right|_{\epsilon/N^2} \leq C_2 \epsilon \ . \tag{1.81}$$

The main point is (Ψ4), in which the estimates of the domain of definition of the map Ψ holds uniformly in the limit $N \to \infty$.

We show now how Theorem 1.3 follows from Kuksin-Perelman Theorem 1.31.

Proof of Theorem 1.3. Introduce the weights $w^1 := \{N^{3/2}[k]_N^s e^{\sigma[k]_N} \omega\left(\frac{k}{N}\right)^{1/2}\}_{k=1}^{N-1}$ and $w^2 := \{N^{3/2}[k]_N^{s+1} e^{\sigma[k]_N} \omega\left(\frac{k}{N}\right)^{1/2}\}_{k=1}^{N-1}$ and consider the map Ψ of Theorem 1.45 as a map from $\mathcal{P}^{w^1}$ in itself. Since for any $(\xi, \eta) \in \mathcal{P}^{w^1}$ one has that

$$\|(\xi,\eta)\|_{\mathcal{P}^{w^1}} \equiv N^2 \|(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}} \ , \tag{1.82}$$

it follows by scaling that there exists a constant $C_3 > 0$, independent of N, such that

$$\left\|\Psi^0\right\|_{\mathcal{A}^{w^2}_{w^1,\rho}} \leq C_3 \rho^2 \ .$$

Thus, for any $\rho \leq \rho_* \equiv \min\left(\frac{2^{-34}}{C_3}, \epsilon_*\right)$, Ψ satisfies condition (1.40). Thus we can apply Theorem 1.31 to the map Ψ, getting the existence of a symplectic real analytic map $\widetilde{\Psi}$ defined on $B^{w^1}(a\rho_*)$ which satisfies $i) - iv)$ of Theorem 1.31.
By Lemma 1.65 the map $\widetilde{\Psi}$ is invertible in $B^{w^1}(\mu a \rho_*)$ and its inverse Φ satisfies $\Phi = \mathbb{1} + \Phi^0$ with $\Phi^0 \in \mathcal{A}^{w^2}_{w^1, \mu a \rho_*}$. To get the statement of the theorem simply reexpress the map Φ in terms of real variables (x,y), (X,Y) and denote such a map by Φ_N. □

Remark 1.46. *By the proof of Theorem 1.3 above one deduces the estimate*

$$\sup_{\|(\phi,\psi)\|_{\mathcal{P}^{s,\sigma}} \leq R_{s,\sigma}/N^2} \left\|d\Phi^0(\phi,\psi)^*\right\|_{\mathcal{L}(\mathcal{P}^{s,\sigma}, \mathcal{P}^{s+1,\sigma})} \leq C_{s,\sigma} R_{s,\sigma} \ , \tag{1.83}$$

for some $C_{s,\sigma} > 0$*, independent of* N*.*

The rest of this subsection is devoted to the proof of Theorem 1.45.
In the following it will be convenient to consider the variables (b,a) defined in (1.75) dropping the conditions $\sum_{j=0}^{N-1} b_j = 0$ and $\prod_{j=0}^{N-1}(1+a_j) = 1$. Equation (1.76) suggests to introduce on the variables b, a the norm

$$\|(b,a)\|^2_{\mathcal{C}^{s,\sigma}} := \frac{1}{2N} \sum_{k=0}^{N-1} \max(1, [k]_N^{2s}) e^{2\sigma[k]_N} \left(|\hat{b}_k|^2 + 4|\hat{a}_k|^2\right) \tag{1.84}$$

and to define the space

$$\mathcal{C}^{s,\sigma}_{\mathbb{R}} := \left\{(b,a) \in \mathbb{R}^N \times \mathbb{R}^N \ : \ \|(b,a)\|_{\mathcal{C}^{s,\sigma}} < \infty\right\} \ . \tag{1.85}$$

We will write $\mathcal{C}^{s,\sigma}$ for the complexification of $\mathcal{C}^{s,\sigma}_{\mathbb{R}}$.

In the following we will consider normally analytic map between the spaces $\mathcal{P}^{s,\sigma}$ and $\mathcal{C}^{s,\sigma}$. We need to specify the basis of $\mathcal{C}^{s,\sigma}$ that we will use to verify the property of being normally analytic. While it is quite hard to verify this property when the basis is general, it turns out that it is quite easy to verify it using the basis of complex exponentials defined in (1.3). Indeed the norm (1.84) is given in term of the Fourier variables. For the same reason, it will be convenient to express a map from $\mathcal{C}^{s,\sigma}$ to $\mathcal{P}^{s,\sigma}$ as a function of the Fourier variables $\hat{b}$, $\hat{a}$.

We prove now some analytic properties of the map Θ defined in (1.75). In the following we will denote by Θ_Ξ the map Θ expressed in the (ξ,η) variables.

Proposition 1.47. *The map Θ_Ξ satisfies the following properties:*

(Θ1) $\Theta_\Xi(0,0)=(0,0)$. *Furthermore let $d\Theta_\Xi(0,0)$ be the linearization of Θ_Ξ at $(\xi,\eta)=(0,0)$. Then $(B,A)=d\Theta_\Xi(0,0)[(\xi,\eta)]$ iff*

$$\begin{aligned} &\widehat{B}_0=0, \qquad \widehat{B}_k=-\left(\tfrac{1}{2}\omega\left(\tfrac{k}{N}\right)\right)^{1/2}(\xi_k+\eta_{N-k}), \qquad 1\le k\le N-1\ ,\\ &\widehat{A}_0=0, \qquad \widehat{A}_k=-\mathrm{i}\varpi_k\left(2\omega\left(\tfrac{k}{N}\right)\right)^{-1/2}(\xi_k-\eta_{N-k}), \qquad 1\le k\le N-1. \end{aligned} \tag{1.86}$$

where $\varpi_k:=(1-e^{-2\mathrm{i}\pi k/N})/2$, $\forall\, 1\le k\le N-1$.

Moreover for any $s\ge 0$, $\sigma\ge 0$ there exist constants $C_{\Theta_1},C_{\Theta_2}>0$, independent of N, such that

$$\left\|\underline{d\Theta_\Xi}(0,0)\right\|_{\mathcal{L}(\mathcal{P}^{s,\sigma},\mathcal{C}^{s,\sigma})}\le C_{\Theta_1}, \qquad \left\|\underline{d\Theta_\Xi}(0,0)^*\right\|_{\mathcal{L}(\mathcal{C}^{s+2,\sigma},\mathcal{P}^{s+1,\sigma})}\le \frac{C_{\Theta_2}}{N}\ . \tag{1.87}$$

(Θ2) *Let $\Theta^0_\Xi:=\Theta_\Xi-d\Theta_\Xi(0,0)$. For any $s\ge 0$, $\sigma\ge 0$, there exist constants $C_{\Theta_3},C_{\Theta_4},\epsilon_*>0$, independent of N, such that the map $\Theta^0_\Xi\in\mathcal{N}_{\epsilon_*/N^2}(\mathcal{P}^{s,\sigma},\mathcal{C}^{s+1,\sigma})$ and the map $[d\Theta^0_\Xi]^*\in\mathcal{N}_{\epsilon_*/N^2}(\mathcal{P}^{s,\sigma},\,\mathcal{L}(\mathcal{C}^{s+2,\sigma},\,\mathcal{P}^{s+1,\sigma}))$, and*

$$\begin{aligned} \left|\underline{\Theta^0_\Xi}\right|_{\epsilon/N^2} &\equiv \sup_{\|(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}}\le\epsilon/N^2}\left\|\underline{\Theta^0_\Xi}(\xi,\eta)\right\|_{\mathcal{C}^{s+1,\sigma}}\le\frac{C_{\Theta_3}\epsilon^2}{N^2};\\ \left|\underline{[d\Theta^0_\Xi]^*}\right|_{\epsilon/N^2} &\equiv \sup_{\|(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}}\le\epsilon/N^2}\left\|\underline{d\Theta^0_\Xi}(\xi,\eta)^*\right\|_{\mathcal{L}(\mathcal{C}^{s+2,\sigma},\,\mathcal{P}^{s+1,\sigma})}\le\frac{C_{\Theta_4}\epsilon}{N^2}. \end{aligned} \tag{1.88}$$

The proof of the proposition is postponed in Appendix C. Note that the estimates (1.87) and (1.88) imply that there exists a constant $C_{\Theta_5}>0$, independent of N, such that for any $\rho\le\frac{\epsilon_*}{N^2}$ one has $\Theta_\Xi\in\mathcal{N}_\rho(\mathcal{P}^{s,\sigma},\mathcal{C}^{s,\sigma})$ and

$$\left|\underline{\Theta_\Xi}\right|_\rho\le C_{\Theta_5}\,\rho\ . \tag{1.89}$$

We start now the perturbative construction of the Birkhoff coordinates for the Toda lattice, which is based on the construction of the spectrum and of the eigenfunctions of $L_{b,a}$ (defined in (1.78)) as a perturbation of the free operator $L_0:=L_{b,a}|_{(b,a)=(0,0)}$. More precisely we decompose $L_{b,a}=L_0+L_p$,

where

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & \dots & 1 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & \dots & 1 & 0 \end{pmatrix}, \quad L_p = \begin{pmatrix} b_0 & a_0 & 0 & \dots & a_{N-1} \\ a_0 & b_1 & a_1 & \ddots & \vdots \\ 0 & a_1 & b_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-2} \\ a_{N-1} & \dots & \dots & a_{N-2} & b_{N-1} \end{pmatrix} \tag{1.90}$$

and following the approach in [KP10, BGGK93, Kap91] we apply Kato perturbation theory [Kat66]. The next lemma characterizes completely the spectrum of L_0 as an operator on $\mathbb{C}^{2N}$:

Lemma 1.48. *Consider L_0 as an operator on $\mathbb{C}^{2N}$, then its eigenvalues and normalized eigenvectors are:*

eigenvalues	*eigenvectors*	
$\lambda_0^0 = -2,$	$f_{00}(k) = \frac{1}{\sqrt{2N}}(-1)^k$	
$\lambda_{2j-1}^0 = \lambda_{2j}^0 = -2\cos\left(\frac{j\pi}{N}\right),$	$f_{2j-1,0}(k) = \frac{1}{\sqrt{2N}} e^{-\mathrm{i}\rho_j k},\ f_{2j,0}(k) = \frac{1}{\sqrt{2N}} e^{\mathrm{i}\rho_j k},$	$1 \le j \le N-1$
$\lambda_{2N-1}^0 = 2,$	$f_{2N-1,0}(k) = \frac{1}{\sqrt{2N}}$	

where $0 \le k \le 2N-1$ and $\rho_j := \left(1 + \frac{j}{N}\right)\pi$. In particular the gaps of L_0 are all closed.

The proof is an easy computation and can be found in [HK08b].

Remark 1.49. *For $0 \le j,k \le \lfloor N/2 \rfloor$ one has $\left|\lambda_{2j}^0 - \lambda_{2k}^0\right|,\ \left|\lambda_{2N-j}^0 - \lambda_{2N-k}^0\right| \ge \frac{4|j^2-k^2|}{N^2}$. In particular if $j \ne k$ then $\left|\lambda_{2j}^0 - \lambda_{2k}^0\right| \ge 1/N^2$.*

We use now Kato perturbation theory of operators in order to introduce the main objects needed in the following and to give some preliminary estimates.

For $1 \le j \le N-1$ let $E_j(b,a)$ be the two-dimensional subspace spanned by the eigenvectors corresponding to the eigenvalues $\lambda_{2j-1}(b,a)$ and $\lambda_{2j}(b,a)$ of $L_{b,a}$. Analogously, let $E_0(b,a)$ (respectively $E_N(b,a)$) be the one-dimensional subspace spanned by the eigenvector of $\lambda_0(b,a)$ (respectively $\lambda_{2N-1}(b,a)$). Introduce the spectral projector on $E_j(b,a)$ defined by

$$P_j(b,a) = -\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_j} (L_{b,a} - \lambda)^{-1}\,\mathrm{d}\lambda, \qquad 0 \le j \le N \tag{1.91}$$

where, for $1 \le j \le N-1$, Γ_j is a closed path counter-clockwise oriented in $\mathbb{C}$ which encloses the eigenvalues $\lambda_{2j-1}(b,a)$ and $\lambda_{2j}(b,a)$ and does not contain any other eigenvalue of $L_{b,a}$. Analogously, Γ_0 (respectively Γ_N) encloses the eigenvalue $\lambda_0(b,a)$ (respectively $\lambda_{2N-1}(b,a)$) and no other eigenvalue of $L_{b,a}$. $P_j(b,a)$ maps $\mathbb{C}^{2N}$ onto $E_j(b,a)$ and, as we will prove, is well defined for (b,a) small enough. $P_j(0,0)$ will be denoted by P_{j0} and its range $E_j(0,0)$, which will be denoted by E_{j0}, is given by

$$\mathrm{Im}\, P_{j0} = E_{j0},\ E_{j0} = \mathrm{span}\,\langle f_{2j,0}, f_{2j-1,0}\rangle\,.$$

Define also the transformation operators

$$U_j(b,a) = \left(\mathbb{1} - (P_j(b,a) - P_{j0})^2\right)^{-1/2} P_j(b,a), \quad 1 \le j \le N-1. \tag{1.92}$$

U_j has the property of mapping isometrically E_{j0} into the subspace $E_j(b,a)$ spanned by the perturbed eigenvectors [Kat66]. Remark, however, that in general the image of an unperturbed eigenvector is *not* an eigenvector itself. We prove now some properties of the just defined objects.

Lemma 1.50. *There exist a constant $C_{s,\sigma} > 0$, independent of N, such that the map $(b,a) \mapsto L_p(b,a)$ is analytic as a map from $\mathcal{C}^{s,\sigma}$ to $\mathcal{L}\left(\mathbb{C}^{2N}\right)$. Moreover*

$$\|L_p(b,a)\|_{\mathcal{L}(\mathbb{C}^{2N})} \leq C_{s,\sigma} \|(b,a)\|_{\mathcal{C}^{s,\sigma}} . \tag{1.93}$$

Then by Kato theory one has the corollary

Corollary 1.51. *There exist constants $C_{s,\sigma}$, $\epsilon_* > 0$, independent of N, such that the following holds true:*

(i) The spectrum of $L_{b,a}$ is close to the spectrum of L_0; in particular for any $(b,a) \in B^{\mathcal{C}^{s,\sigma}}\left(\frac{\epsilon_*}{N^2}\right)$

$$\left|\lambda_{2j}(b,a) - \lambda_{2j}^0\right|, \left|\lambda_{2j-1}(b,a) - \lambda_{2j-1}^0\right| \leq C_{s,\sigma} \|(b,a)\|_{\mathcal{C}^{s,\sigma}} . \tag{1.94}$$

(ii) One has that $(b,a) \mapsto P_j(b,a)$ is analytic as a map from $B^{\mathcal{C}^{s,\sigma}}\left(\frac{\epsilon_*}{N^2}\right)$ to $\mathcal{L}(\mathbb{C}^{2N})$. Moreover for $(b,a) \in B^{\mathcal{C}^{s,\sigma}}\left(\frac{\epsilon_*}{N^2}\right)$ one has

$$\|P_j(b,a) - P_{j0}\|_{\mathcal{L}(\mathbb{C}^{2N})} \leq C_{s,\sigma} \|(b,a)\|_{\mathcal{C}^{s,\sigma}} . \tag{1.95}$$

(iii) For each $1 \leq j \leq N-1$, the maps U_j, defined in (1.92), are well defined from $B^{\mathcal{C}^{s,\sigma}}\left(\frac{\epsilon_*}{N^2}\right)$ to $\mathcal{L}(\mathbb{C}^{2N})$ and satisfy the following algebraic properties:

- *$(U1)$ $Im\, U_j(b,a) = E_j(b,a)$;*
- *$(U2)$ for (b,a) real, one has $\overline{U_j(b,a)f} = U_j(b,a)\bar{f}$;*
- *$(U3)$ for (b,a) real and $f \in E_{j0}$, one has $\|U_j(b,a)f\|_{\mathbb{C}^{2N}} = \|f\|_{\mathbb{C}^{2N}}$.*

Finally the following analytic property holds:

- *$(U4)$ One has that $(b,a) \mapsto U_j(b,a)$ is analytic as a map from $B^{\mathcal{C}^{s,\sigma}}\left(\frac{\epsilon_*}{N^2}\right)$ to $\mathcal{L}(\mathbb{C}^{2N})$. Moreover for $(b,a) \in B^{\mathcal{C}^{s,\sigma}}\left(\frac{\epsilon_*}{N^2}\right)$ one has*

$$\|U_j(b,a) - P_j(b,a)\|_{\mathcal{L}(\mathbb{C}^{2N})} \leq C_{s,\sigma} \|(b,a)\|_{\mathcal{C}^{s,\sigma}}^2 . \tag{1.96}$$

The proofs of Lemma 1.50 and Corollary 1.51 can be found in Appendix D.

For $1 \leq j \leq N-1$ and $(b,a) \in B^{\mathcal{C}^{s,\sigma}}\left(\frac{\epsilon_*}{N^2}\right)$ define now the vectors

$$f_{2j-1}(b,a) := U_j(b,a) f_{2j-1,0}, \qquad \text{and} \qquad f_{2j}(b,a) := U_j(b,a) f_{2j,0} \tag{1.97}$$

which by property $(U1)$ belong to $E_j(b,a)$. Define also the maps

$$\begin{aligned} z_j(b,a) &:= \left(\tfrac{2}{N}\omega\left(\tfrac{j}{N}\right)\right)^{-1/2} \left\langle \left(L_{b,a} - \lambda_{2j}^0\right) f_{2j}(b,a), \overline{f_{2j}(b,a)} \right\rangle, \\ w_j(b,a) &:= \left(\tfrac{2}{N}\omega\left(\tfrac{j}{N}\right)\right)^{-1/2} \left\langle \left(L_{b,a} - \lambda_{2j-1}^0\right) f_{2j-1}(b,a), \overline{f_{2j-1}(b,a)} \right\rangle \end{aligned} \tag{1.98}$$

where $\langle u, v\rangle = \sum u_j \overline{v_j}$ is the Hermitian product in $\mathbb{C}^{2N}$. Finally denote $z(b,a) = (z_1(b,a), \ldots, z_{N-1}(b,a))$ and $w(b,a) = (w_1(b,a), \ldots, w_{N-1}(b,a))$, and let Z be the map

$$(b,a) \mapsto Z(b,a) := (z(b,a), w(b,a)). \tag{1.99}$$

The map Ψ of Theorem 1.45 will be constructed by expressing Z as a function of the linear Birkhoff coordinates ξ, η.

The properties of the map Z are collected in the next lemma which constitutes the main technical step for the application of Kuksin-Perelman Theorem to the Toda lattice.

Lemma 1.52. *The map Z, defined by (1.99), is well defined for $(b,a) \in B^{\mathcal{C}^{s,\sigma}}\left(\frac{\epsilon_*}{N^2}\right)$. If b, a are real valued and fulfill $\|(b,a)\|_{\mathcal{C}^{s,\sigma}} \leq \frac{\epsilon_*}{N^2}$, then, for every $1 \leq j \leq N-1$, the following properties are also fulfilled:*

$(Z1)$ $\overline{z_j(b,a)} = w_j(b,a)$;

$(Z2)$ $\gamma_j^2 = \frac{2}{N}\omega\left(\frac{j}{N}\right)|z_j(b,a)|^2 = \frac{2}{N}\omega\left(\frac{j}{N}\right)|w_j(b,a)|^2$;

$(Z3)$ $z_j(0,0) = w_j(0,0) = 0$; *moreover the linearizations of z_j and w_j at $(b,a) = (0,0)$ are given by*

$$\begin{aligned} dz_j(0,0)[(B,A)] &= \left(2\omega\left(\tfrac{j}{N}\right)\right)^{-1/2}\left(\hat{B}_j - 2e^{\mathrm{j}i\pi/N}\hat{A}_j\right), \\ dw_j(0,0)[(B,A)] &= \left(2\omega\left(\tfrac{j}{N}\right)\right)^{-1/2}\left(\hat{B}_{N-j} - 2e^{-\mathrm{j}i\pi/N}\hat{A}_{N-j}\right). \end{aligned} \tag{1.100}$$

The map $dZ(0,0) = (dz(0,0), dw(0,0))$ is in the class $\mathcal{L}(\mathcal{C}^{s,\sigma}, \mathcal{P}^{s,\sigma})$. Its adjoint $dZ(0,0)^*$ is in the class $\mathcal{L}(\mathcal{P}^{s,\sigma}, \mathcal{C}^{s+1,\sigma})$. Finally there exist constants $C_{Z_1}, C_{Z_2} > 0$, independent of N, such that for any $s \geq 0$ and $\sigma \geq 0$

$$\|\underline{dZ}(0,0)\|_{\mathcal{L}(\mathcal{C}^{s,\sigma},\, \mathcal{P}^{s,\sigma})} \leq C_{Z_1}, \quad \|\underline{dZ}(0,0)^*\|_{\mathcal{L}(\mathcal{P}^{s,\sigma},\, \mathcal{C}^{s+2,\sigma})} \leq C_{Z_2}N^2 \ . \tag{1.101}$$

$(Z4)$ *For any $s \geq 0$, $\sigma \geq 0$, there exist constants $C_{Z_3}, C_{Z_4}, \epsilon_* > 0$, independent of N, such that for every $0 < \epsilon \leq \epsilon_*$ the map $Z^0 := Z - dZ(0,0) \in \mathcal{N}_{\epsilon/N^2}\left(\mathcal{C}^{s,\sigma}, \mathcal{P}^{s+1,\sigma}\right)$ and the map $[dZ^0]^* \in \mathcal{N}_{\epsilon/N^2}\left(\mathcal{C}^{s,\sigma}, \mathcal{L}(\mathcal{P}^{s,\sigma}, \mathcal{C}^{s+2,\sigma})\right)$. Moreover*

$$\begin{aligned} \sup_{\|(b,a)\|_{\mathcal{C}^{s,\sigma}} \leq \epsilon/N^2} \left\|\underline{Z^0}(b,a)\right\|_{\mathcal{P}^{s+1,\sigma}} &\leq C_{Z_3}\frac{\epsilon^2}{N^2}, \\ \sup_{\|(b,a)\|_{\mathcal{C}^{s,\sigma}} \leq \epsilon/N^2} \left\|\underline{dZ^0}(b,a)^*\right\|_{\mathcal{L}(\mathcal{P}^{s,\sigma},\, \mathcal{C}^{s+2,\sigma})} &\leq C_{Z_4}N\epsilon. \end{aligned} \tag{1.102}$$

The proof of the lemma is very technical, and is postponed in Appendix E.

Remark 1.53. *In the limit of infinitely many particles, the linearization $dz_j(0,0)(b,a)$ at the different edges of the spectrum are given by*

$$dz_j(0,0)(B,A) \approx \frac{\hat{B}_j - 2\hat{A}_j}{\sqrt{2\omega(j/N)}} \quad \text{if } j/N \ll 1 \qquad dz_j(0,0)(B,A) \approx \frac{\hat{B}_j + 2\hat{A}_j}{\sqrt{2\omega(j/N)}} \quad \text{if } 1 - j/N \ll 1 \ . \tag{1.103}$$

The existence of two different sequences is in agreement with the works [BKP13b, BKP13a], in which the spectrum of the Lax operator associated to the Toda lattice is approximated, up to a small error, by the spectrum of two Sturm-Liouville operators associated to two KdV equations. More explicitly, in [BKP13b] the following result is proved: take $\alpha, \beta \in C^\infty(\mathbb{T})$ such that $\int_{\mathbb{T}} \alpha = \int_{\mathbb{T}} \beta = 0$, $a_j = 1 + \frac{1}{N^2}\alpha(j/N)$ and $b_j = \frac{1}{N^2}\beta(j/N)$. Then the spectrum of the Lax matrix (1.78) with a_j, b_j as elements can be approximated at the two edges by the spectrum of the two Sturm-Liouville operators $L = -\frac{d^2}{dx^2} + (\beta \pm 2\alpha)$ on $C^\infty(\mathbb{T})$.

We are ready to define the map Ψ of Theorem 1.45: let

$$\Psi : \mathcal{P}^{s,\sigma} \to \mathcal{P}^{s,\sigma}, \qquad (\xi,\eta) \mapsto (\phi(\xi,\eta), \psi(\xi,\eta)) \tag{1.104}$$

defined by

$$\Psi = -Z \circ \Theta_\Xi; \quad \text{i. e.} \quad \phi = -z \circ \Theta_\Xi, \qquad \psi = -w \circ \Theta_\Xi. \tag{1.105}$$

We show now that Ψ satisfies the properties $(\Psi1) - (\Psi4)$ claimed in Theorem 1.45.

Proof of Theorem 1.45. Property $(\Psi1)$ and $(\Psi2)$ follows by $(Z1)$ respectively $(Z2)$. We prove now $(\Psi3)$. By $(\Theta1)$ and $(Z3)$ one has $\Psi(0,0) = (0,0)$. In order to compute $d\Psi(0,0) = (d\phi(0,0), d\psi(0,0))$ note that

$$d\phi(0,0) = -dz(0,0)\, d\Theta_\Xi(0,0) = -(dz(0,0)\mathcal{F}^{-1}) \circ (\mathcal{F} d\Theta_\Xi(0,0)) \ .$$

Let $(\hat{B}, \hat{A}) = \mathcal{F} d\Theta_\Xi(0,0)(\xi,\eta)$. Then (1.100) and (1.86) imply that, for $1 \leq j \leq N-1$,

$$\begin{aligned} d\phi_j(0,0)(\xi,\eta) &= -\frac{1}{\sqrt{2\omega(j/N)}} \left(\hat{B}_j - 2e^{\mathrm{i}\pi j/N} \hat{A}_j \right) \\ &= \frac{1}{\sqrt{2\omega(j/N)}} \left(\sqrt{\frac{\omega(j/N)}{2}} (\xi_j + \eta_{N-j}) - \mathrm{i} \frac{2e^{\mathrm{i}\pi j/N} \varpi_j}{\sqrt{2\omega(j/N)}} (\xi_j - \eta_{N-j}) \right) \equiv \xi_j \ , \end{aligned}$$

where we used that $2e^{\mathrm{i}\pi j/N}\varpi_j = \mathrm{i}\omega\left(\frac{j}{N}\right)$. One verifies analogously that $d\psi_j(0,0)(\xi,\eta) = \eta_j$.

We prove now property $(\Psi4)$, which is a consequence of the fact that the space of normally analytic maps is closed by composition (see Lemma 1.63). Fix $s \geq 0$ and $\sigma \geq 0$. Let $0 < \epsilon \leq \frac{\epsilon_*}{C_{\Theta_5}}$, where C_{Θ_5} is the constant in (1.89). Since $Z = dZ(0,0) + Z^0$ and $\Theta_\Xi = d\Theta_\Xi(0,0) + \Theta^0_\Xi$, one gets that

$$\Psi^0 = -Z^0 \circ \Theta_\Xi - dZ(0,0) \circ \Theta^0_\Xi \ . \tag{1.106}$$

Thus properties $(Z3)$, $(\Theta2)$ and estimate (1.89) imply that there exists a constant $C > 0$, independent of N, such that

$$\left|\underline{\Psi^0}\right|_{\epsilon/N^2} \equiv \sup_{\|(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}} \leq \epsilon/N^2} \left\|\underline{\Psi^0}(\xi,\eta)\right\|_{\mathcal{P}^{s+1,\sigma}} \leq \frac{C\,\epsilon^2}{N^2} \ ,$$

which proves the first estimate of $(\Psi4)$. We study now the adjoint map $d\Psi^0(\xi,\eta)^*$. Writing $d\Theta_\Xi = d\Theta_\Xi(0,0) + d\Theta^0_\Xi$ one gets that

$$\begin{aligned} d\Psi^0(\xi,\eta)^* &= -d\Theta_\Xi(0,0)^*\, dZ^0(\Theta_\Xi(\xi,\eta))^* - d\Theta^0_\Xi(\xi,\eta)^*\, dZ^0(\Theta_\Xi(\xi,\eta))^* - d\Theta^0_\Xi(\xi,\eta)^*\, dZ(0,0)^* \\ &= I + II + III. \end{aligned}$$

We estimate each term in the expression displayed above. In the following, if $A \in \mathcal{N}_\rho(\mathcal{P}^{s,\sigma}, \mathcal{L}(\mathcal{P}^{s,\sigma}, \mathcal{P}^{s+1,\sigma}))$, we denote by

$$|\underline{A}|_\rho \equiv \sup_{\|(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}} \leq \epsilon/N^2} \|\underline{A}(\xi,\eta)\|_{\mathcal{L}(\mathcal{P}^{s,\sigma},\, \mathcal{P}^{s+1,\sigma})} \, .$$

We begin by estimating I:

$$|\underline{I}|_{\epsilon/N^2} \leq \frac{C_{\Theta_2}}{N} \sup_{\|(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}} \leq \epsilon/N^2} \left\|\underline{dZ^0}(\underline{\Theta_\Xi}(\xi,\eta))^*\right\|_{\mathcal{L}(\mathcal{P}^{s,\sigma},\, \mathcal{C}^{s+2,\sigma})} \leq \frac{C_{\Theta_2}}{N} C_{Z_4} C_{\Theta_5} N\,\epsilon \leq C\epsilon,$$

where in the first inequality we used the second estimate of (1.87) and in the second inequality we used the second estimate in (1.102). Now we study II:

$$|\underline{II}|_{\epsilon/N^2} \leq \frac{C_{\Theta_4}\epsilon}{N^2} \sup_{\|(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}} \leq \epsilon/N^2} \left\| \underline{dZ^0}(\underline{\Theta_\Xi}(\xi,\eta))^* \right\|_{\mathcal{L}(\mathcal{P}^{s,\sigma},\, \mathcal{C}^{s+2,\sigma})} \leq \frac{C_{\Theta_4}\epsilon}{N^2} C_{Z_4} C_{\Theta_5} N\epsilon \leq \frac{C\epsilon^2}{N},$$

where we used the second estimate in (1.88) and again $(Z4)$. Finally, using again $(\Theta 2)$ and the second estimate of (1.101), one has

$$|\underline{III}|_{\epsilon/N^2} \leq \frac{C_{\Theta_4}\epsilon}{N^2} \left\| \underline{dZ}(0,0)^* \right\|_{\mathcal{L}(\mathcal{P}^{s,\sigma},\, \mathcal{C}^{s+2,\sigma})} \leq \frac{C_{\Theta_4}\epsilon}{N^2} C_{Z_2} N^2 \leq C\epsilon \ .$$

Collecting the estimates above one gets

$$\left|[\underline{d\Psi^0}]^*\right|_{\epsilon/N^2} \equiv \sup_{\|(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}} \leq \epsilon/N^2} \left\| \underline{d\Psi^0}(\xi,\eta)^* \right\|_{\mathcal{L}(\mathcal{P}^{s,\sigma},\, \mathcal{P}^{s+1,\sigma})} \leq 3C\epsilon,$$

and $(\Psi 4)$ follows. □

Proof of Corollary 1.6. Provided $0 < R < R'_{s,\sigma}$ is small enough, one has that $w_0 := \Phi_N^{-1}(v_0)$ fulfills

$$\|w_0\|_{\mathcal{P}^{s,\sigma}} \leq \frac{R}{N^2}(1 + CR) \ ,$$

and, denoting by $w(t)$ the solution in Birkhoff coordinates, one has $\|w_0\|_{\mathcal{P}^{s,\sigma}} = \|w(t)\|_{\mathcal{P}^{s,\sigma}}$. Thus, provided $0 < R < R'_{s,\sigma}$ is small enough one has

$$\|v(t)\|_{\mathcal{P}^{s,\sigma}} = \|\Phi_N(w(t))\|_{\mathcal{P}^{s,\sigma}} \leq \frac{R}{N^2}(1 + C'R)$$

which implies the thesis. □

3.2 Proof of Theorem 1.7

The proof is based on the construction of the first terms of the Taylor expansion of Φ_N through Birkhoff normal form. To this end we work with the complex variables (ξ,η) (defined in (1.25)) and will eventually restrict to the real subspace $\mathcal{P}^{s,\sigma}_{\mathbb{R}}$.

Remark 1.54. *Consider the Taylor expansion of Φ_N at the origin, one has*

$$\Phi_N = \mathbb{1} + Q^{\Phi_N} + O(\|(\xi,\eta)\|^3_{\mathcal{P}^{s,\sigma}}) \ ,$$

then Q^{Φ_N} is a bounded quadratic polynomial. Furthermore, since Φ_N is canonical, Q^{Φ_N} is a Hamiltonian vector field, i.e. there exists a cubic complex valued polynomial χ_{Φ_N} s.t. Q^{Φ_n} is the Hamiltonian vector field of χ_{Φ_N}.

We need a preliminary result about a uniqueness property of the transformation introducing Birkhoff coordinates (called below Birkhoff map).

Lemma 1.55. *Let Φ_N and Ψ_N be Birkhoff maps for H_{Toda}, analytic in some neighborhood of the origin; assume that $d\Phi_N(0,0) \equiv d\Psi_N(0,0) = \mathbb{1}$ and denote by χ_{Φ_N} and χ_{Ψ_N} the Hamiltonian functions corresponding to Q^{Φ_N} and Q^{Ψ_N} respectively, then one has*

$$\{H_0; \chi_{\Phi_N} - \chi_{\Psi_N}\} = 0 \ , \tag{1.107}$$

where H_0 is defined in (1.6).

Proof. By a standard computation of the Taylor expansion one has

$$H_{Toda} \circ \Phi_N = H_0 + \{H_0, \chi_{\Phi_N}\} + H_1 + h.o.t.$$

where H_1 is the function

$$H_1(q) = \sum_{j=0}^{N-1} \frac{(q_j - q_{j+1})^3}{6}$$

Since Φ_N is a Birkhoff map, the function $H_{Toda} \circ \Phi_N$ is in Birkhoff normal form so in particular its Taylor expansion contains only terms of even degree. Thus the cubic terms in the expansion above must vanish: $\{H_0, \chi_{\Phi_N}\} + H_1 = 0$. The same argument holds also for the map Ψ_N, thus the thesis follows. $\square$

Remark 1.56. *Writing as usual*

$$\chi_{\Phi_N}(\xi, \eta) = \sum_{|K|+|L|=3} \chi_{K,L} \xi^K \eta^L \ ,$$

one gets that, since

$$\{H_0, \chi_{\Phi_N}\} = -\sum_{|K|+|L|=3} \mathrm{i}\omega \cdot (K-L) \chi_{K,L}\, \xi^K \eta^L \ ,$$

eq. (1.107) *implies that, if for some K, L one has $\omega \cdot (K-L) \neq 0$, then $\chi_{K,L}$ is unique and coincides with $\frac{H_{K,L}}{\mathrm{i}\omega\cdot(K-L)}$ with an obvious definition of $H_{K,L}$.*

Lemma 1.57. *In terms of the variables (ξ, η) one has*

$$H_1(\xi,\eta) = \frac{1}{12\sqrt{2N}} \left[\sum_{\substack{k_1+k_2+k_3=0 \bmod N \\ 1\le k_1,k_2,k_3,\le N-1}} (-1)^{\frac{k_1+k_2+k_3}{N}} \sqrt{\omega_{k_1}}\sqrt{\omega_{k_2}}\sqrt{\omega_{k_3}} \left(\xi_{k_1}\xi_{k_2}\xi_{k_3} + \eta_{k_1}\eta_{k_2}\eta_{k_3}\right) \right.$$
$$\left. + 3 \sum_{\substack{k_1+k_2-k_3=0 \bmod N \\ 1\le k_1,k_2,k_3\le N-1}} (-1)^{\frac{k_1+k_2-k_3}{N}} \sqrt{\omega_{k_1}}\sqrt{\omega_{k_2}}\sqrt{\omega_{k_3}} \left(\xi_{k_1}\xi_{k_2}\eta_{k_3} + \eta_{k_1}\eta_{k_2}\xi_{k_3}\right) \right]$$

Proof. First remark that

$$q_j - q_{j+1} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{q}_k \left(1 - e^{-\frac{2\pi \mathrm{i} k}{N}}\right) e^{-\frac{2\pi \mathrm{i} jk}{N}} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathrm{i}\omega_k e^{-\frac{\mathrm{i}\pi k}{N}} \hat{q}_k e^{-\frac{2\pi \mathrm{i} jk}{N}} \ ,$$

so that

$$\frac{1}{6}\sum_{j=0}^{N-1}(q_j-q_{j+1})^3 = \frac{\mathrm{i}^3}{6N^{3/2}}\sum_{k_1,k_2,k_3}\omega_{k_1}\hat{q}_{k_1}\omega_{k_2}\hat{q}_{k_2}\omega_{k_3}\hat{q}_{k_3}e^{-\frac{\mathrm{i}\pi}{N}(k_1+k_2+k_3)}\sum_{j=0}^{N-1}e^{\frac{2\pi\mathrm{i}j}{N}(k_1+k_2+k_3)}$$
$$= \frac{\mathrm{i}^3}{6N^{1/2}}\sum_{k_1+k_2+k_3=0 \bmod N}(-1)^{\frac{k_1+k_2+k_3}{N}}\omega_{k_1}\hat{q}_{k_1}\omega_{k_2}\hat{q}_{k_2}\omega_{k_3}\hat{q}_{k_3} \ .$$

Substituting

$$\omega_k\hat{q}_k = \sqrt{\omega_k}\frac{\xi_k-\eta_{N-k}}{\mathrm{i}\sqrt{2}}$$

and reorganizing the terms one gets the thesis. □

Lemma 1.58. *For any $s \geq 0$, $\sigma \geq 0$, there exists $C > 0$ s.t. one has*

$$\left\|Q^{\phi_N}(\bar{v})\right\|_{\mathcal{P}^{s,\sigma}} \geq CN^2\left\|\bar{v}\right\|^2_{\mathcal{P}^{s,\sigma}} \ , \tag{1.108}$$

where $\bar{v} = ((\xi_1,0,0,...,0),(\bar{\xi}_1,0,0,...,0)) \in \mathcal{P}^{s,\sigma}_{\mathbb{R}}$.

Proof. In this proof, for clarity we denote $\eta_1 := \bar{\xi}_1$, and similarly for the other variables. We are going to compute the ξ_2 component $[Q^{\Phi_N}(\bar{v})]_{\xi_2}$ of $Q^{\Phi_N}(\bar{v})$ and exploit the inequality

$$\left\|Q^{\Phi_N}(\bar{v})\right\|_{\mathcal{P}^{s,\sigma}} \geq \frac{1}{\sqrt{N}}2^s e^{\sigma 2}\omega_2^{1/2}\frac{1}{\sqrt{2}}\left|\left[Q^{\Phi_N}(\bar{v})\right]_{\xi_2}\right| = \frac{2^s e^{\sigma 2}\omega_2^{1/2}}{\sqrt{2N}}\left|\frac{\partial\chi_{\Phi_N}}{\partial\eta_2}(\bar{v})\right| \ ; \tag{1.109}$$

the only monomials in χ_{Φ_N} contributing to such a quantity are quadratic in (ξ_1,η_1) and linear in η_2, but due to the selection rule $k_1 \pm k_2 \pm k_3 = lN$ with a plus for the ξ's and a minus for the η's the only monomial contributing to the r.h.s. of (1.109) is $\chi_{\bar{K},\bar{L}}\xi^{\bar{K}}\eta^{\bar{L}}$ with $\bar{K} := (2,0,...,0)$, and $\bar{L} = (0,1,0,0,...,0)$.

Since

$$\omega\cdot(K-L) = 2\omega_1-\omega_2 = 4\sin\frac{\pi}{N}-2\sin\frac{2\pi}{N} = \frac{2\pi^3}{N^3}+O\left(\frac{1}{N^5}\right) \neq 0 \ , \tag{1.110}$$

such a coefficient is uniquely defined and, for the χ_{Φ_N} corresponding to *any* Birkhoff map, one has

$$\chi_{\bar{K},\bar{L}} = \frac{1}{4\sqrt{2N}}\frac{\omega_1\omega_2^{1/2}}{\mathrm{i}(2\omega_1-\omega_2)} \ . \tag{1.111}$$

Inserting in (1.109) one has that its r.h.s. is equal to

$$\frac{2^s e^{\sigma 2}\omega_2^{1/2}}{\sqrt{2N}}\left|\chi_{\bar{K},\bar{L}}\right|\left|\xi_1\right|^2 = \frac{C''}{N}\frac{\omega_1\omega_2}{|2\omega_1-\omega_2|}\left|\xi_1\right|^2 = C'\frac{\omega_2}{|2\omega_1-\omega_2|}\left\|\bar{v}\right\|^2_{\mathcal{P}^{s,\sigma}} \geq CN^2\left\|\bar{v}\right\|^2_{\mathcal{P}^{s,\sigma}} \ ,$$

where C, C' and C'' are numerical constants independent of N and we used the expansions of ω_1, ω_2 in $1/N$ as well as equation (1.110). □

Proof of Theorem 1.7. The thesis immediately follows taking $\|\bar{v}\|_{\mathcal{P}^{s,\sigma}} = R/N^\alpha$ and imposing the inequality (1.11). □

Proof of Corollary 1.9. By Cauchy inequality and assumption (1.12) Q^{Φ_N} fulfills

$$\left\|Q^{\Phi_N}(\bar{v})\right\|_{\mathcal{P}^{s,\sigma}} \leq \frac{R'}{N^{\alpha'}}\frac{N^{2\alpha}}{R^2}\left\|\bar{v}\right\|^2_{\mathcal{P}^{s,\sigma}} \ . \tag{1.112}$$

Comparing this inequality with (1.108), one gets

$$\frac{R'}{R^2}N^{2\alpha-\alpha'} \geq C''N^2 \ ,$$

which in particular implies the thesis. □

4 FPU packet of modes: proofs.

In this section we prove the results stated in the subsection 1.2 about the persistence of the metastable packet in the FPU system.

To clarify the procedure, we distinguish here between the (ξ,η) variables and the variables (p,q). Thus, we denote by $T:(\xi,\eta)\to(p,q)$ the change of coordinates of the phase space introducing the linear Birkhoff variables (ξ,η) defined in (1.25). Furthermore it is useful to use for the (p,q) variables the following norms

$$\|q\|^2_{s,\sigma} := \frac{1}{N}\sum_{k=0}^{N-1}\max(1,[k]_N^{2s})\,e^{2\sigma[k]_N}\,|\hat{q}_k|^2 \ , \tag{1.113}$$

and

$$\|(p,q)\|_{\mathcal{P}^{s,\sigma}} := \left\|T^{-1}(p,q)\right\|_{\mathcal{P}^{s,\sigma}} \ . \tag{1.114}$$

Lemma 1.59. *Fix $s\geq 1$, $\sigma\geq 0$, then there exist constants $C_1,C_2>0$, independent of N, such that for all $(\xi,\eta)\in\mathcal{P}^{s,\sigma}$ and $\forall l\geq 2$ one has*

$$\|X_{H_l\circ T}(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}} \leq \frac{C_1^l}{(l+1)!}\,\|(\xi,\eta)\|^{l+1}_{\mathcal{P}^{s,\sigma}} \ , \tag{1.115}$$

$$\|X_{H_l\circ T}(\xi,\eta)\|_{\mathcal{P}^{s-1,\sigma}} \leq \frac{C_2^l}{N(l+1)!}\,\|(\xi,\eta)\|^{l+1}_{\mathcal{P}^{s,\sigma}} \,. \tag{1.116}$$

Proof. Define the difference operators by

$$S_\pm : \{q_j\}_{0\leq j\leq N-1}\mapsto\{q_j-q_{j\pm1}\}_{0\leq j\leq N-1} \ , \qquad \text{where } q_N\equiv q_0 \ , \tag{1.117}$$

and the operator $[S_+(q)]^l$ by

$$\left\{[S_+(q)]^l\right\}_j := (q_j-q_{j+1})^l \ ,$$

so that

$$X_{H_l\circ T}(\xi,\eta) = \frac{1}{(l+1)!}T^{-1}\left(\ S_-\left[S_+(T(\xi,\eta))\right]^l,\ 0\right) \ . \tag{1.118}$$

By Lemma 1.70 and Remark 1.72 in Appendix B, there exists a constant $C_{s,\sigma}>0$, independent of N, such that for every integer $n\geq 1$

$$\left\|[S_\pm(q)]^{l+1}\right\|_{s,\sigma} \leq C_{s,\sigma}^{l+1}\,\|S_\pm(q)\|^{l+1}_{s,\sigma} \leq C_{s,\sigma}^{l+1}\,\|(\xi,\eta)\|^{l+1}_{\mathcal{P}^{s,\sigma}} \ , \tag{1.119}$$

where for the last inequality we have identified the couple $(0,q)$ with the corresponding (ξ,η) vector.

Then the thesis follows just remarking that $\left\|T^{-1}(q,0)\right\|_{\mathcal{P}^{s,\sigma}} = \|q\|_{s,\sigma}$, and that S_- is bounded as an operator from $\mathcal{P}^{s,\sigma}$ to itself, while one has

$$\|(S_-(q),0)\|_{\mathcal{P}^{s-1,\sigma}} \le \frac{C}{N}\|q\|_{s,\sigma} \ .$$

□

Introducing the Birkhoff coordinates and using the standard formulae for the pull back of vector fields[3] one has the following

Corollary 1.60. *Fix $s \ge 1$ and $\sigma \ge 0$, then there exist constants $R_{s,\sigma}, C_1, C_2 > 0$, independent of N, such that for all $w \equiv (\phi,\psi) \in B^{s,\sigma}(R_{s,\sigma}/N^2)$ one has*

$$\|X_{H_l\circ T\circ\Phi_N}(w)\|_{\mathcal{P}^{s,\sigma}} \le \frac{C_1^l}{(l+1)!}\|w\|_{\mathcal{P}^{s,\sigma}}^{l+1} \ , \tag{1.120}$$

$$\|X_{H_l\circ T\circ\Phi_N}(w)\|_{\mathcal{P}^{s-1,\sigma}} \le \frac{C_2^l}{N(l+1)!}\|w\|_{\mathcal{P}^{s,\sigma}}^{l+1} \, . \tag{1.121}$$

Remark 1.61. *Write*

$$\tilde{H}_{FPU} \equiv H_{FPU}\circ T\circ\Phi_N = \tilde{H}_{Toda} + \tilde{H}_P \ , \tag{1.122}$$

where

$$\tilde{H}_{Toda} := H_{Toda}\circ T\circ\Phi_N \ , \quad \tilde{H}_P := (\beta-1)H_2\circ T\circ\Phi_N + H^{(3)}\circ T\circ\Phi_N \ , \tag{1.123}$$

then, provided R is small enough the vector field of $\tilde{H}_P$ fulfills the following estimates

$$\left\|X_{\tilde{H}_P}(w)\right\|_{\mathcal{P}^{s,\sigma}} \le C\left[|\beta-1|\,\|w\|_{\mathcal{P}^{s,\sigma}}^3 + C\|w\|_{\mathcal{P}^{s,\sigma}}^4\right] \ , \tag{1.124}$$

$$\left\|X_{\tilde{H}_P}(w)\right\|_{\mathcal{P}^{s-1,\sigma}} \le \frac{C}{N}\left[|\beta-1|\,\|w\|_{\mathcal{P}^{s,\sigma}}^3 + C\|w\|_{\mathcal{P}^{s,\sigma}}^4\right] \ , \tag{1.125}$$

for all $w \in B^{s,\sigma}(R/N^2)$.

In the following we denote by $v(t) \equiv (\xi(t),\bar{\xi}(t))$ the solution of the FPU model in the original Cartesian coordinates (we restrict to the real subspace). We denote by $w(t) := \Phi_N^{-1}(v(t))$ the same solution in Birkhoff coordinates.

Lemma 1.62. *Fix $s \ge 2$ and $\sigma \ge 0$. Then there exist $R'_{s,\sigma}$, T, $C_2 > 0$ such that $v_0 \in B_{\mathbb{R}}^{s,\sigma}\left(\frac{R}{N^2}\right)$ with $R \le R'_{s,\sigma}$ implies $v(t) \in B_{\mathbb{R}}^{s,\sigma}\left(\frac{4R}{N^2}\right)$ for*

$$|t| \le \frac{T}{R^2\mu^4[|\beta-1| + C_2R\mu^2]} \ . \tag{1.126}$$

[3]Namely

$$[\Phi_N^* X](x) = d\Phi_N^{-1}(\Phi_N(x))X(\Phi_N(x))$$

which gives the vector field of the transformed Hamiltonian due to the fact that Φ_N is canonical

Proof. First consider $w_0 := \Phi_N^{-1}(v_0)$ and remark that (provided $R'_{s,\sigma}$ is small enough) one has $w_0 \in B^{s,\sigma}_{\mathbb{R}}\left(\frac{2R}{N^2}\right)$. Denote by $M(w) := \|w\|^2_{\mathcal{P}^{s,\sigma}_{\mathbb{R}}}$. Since $\left\{M, \tilde{H}_{Toda}\right\} \equiv 0$, one has

$$M(w(t)) = M(w_0) + \int_0^t \left\{M; \tilde{H}_P\right\}(w(s))ds \ . \tag{1.127}$$

Denoting $\bar{M}(t) := \sup_{|s|\leq t} M(w(s))$, one has

$$\bar{M}(w(t)) \leq M(w_0) + \int_0^t \left|\left\{M; \tilde{H}_P\right\}(w(s))\right| ds \tag{1.128}$$

$$\leq M(w_0) + \int_0^t \left(C\|w(s)\|^4_{\mathcal{P}^{s,\sigma}_{\mathbb{R}}} |\beta-1| + C\|w(s)\|^5_{\mathcal{P}^{s,\sigma}_{\mathbb{R}}}\right) ds$$

$$\leq M(w_0) + \int_0^t C\bar{M}(t)^2\left(|\beta-1| + C\bar{M}(t)^{1/2}\right) ds$$

$$\leq M(w_0) + |t|C\bar{M}(t)^2\left(|\beta-1| + C\bar{M}(t)^{1/2}\right) \ , \tag{1.129}$$

where, in order to prove the second inequality we used $\left\{M; \tilde{H}_P\right\} := dMX_{\tilde{H}_P}$ and

$$\|dM(w)\|_{\mathcal{L}(\mathcal{P}^{s,\sigma},\mathbb{C})} \leq C\|w\|_{\mathcal{P}^{s,\sigma}} \ ,$$

which follows from an explicit computation. Taking t as in the statement of the Lemma we have that (1.128)-(1.129) ensures $\bar{M}(t) \leq 9M(w(0))/4$, which implies $w(t) \in B^{s,\sigma}_{\mathbb{R}}\left(\frac{3R}{N^2}\right)$ from which the thesis immediately follows. □

Proof of Theorem 1.16. Inequality (1.21) is a direct consequence of Lemma 1.62. To prove inequality (1.22) remark that $\dot{I}_k = \{I_k, \tilde{H}_P\} = x_k\frac{\partial \tilde{H}_P}{\partial y_k} - y_k\frac{\partial \tilde{H}_P}{\partial x_k}$. Thus

$$\frac{1}{N}\sum_{k=1}^{N-1}[k]_N^{2s-2}e^{2\sigma[k]_N}\omega\left(\tfrac{k}{N}\right)\left|\{I_k, \tilde{H}_P\}\right| = \frac{1}{N}\sum_{k=1}^{N-1}[k]_N^{2s-2}e^{2\sigma[k]_N}\omega\left(\tfrac{k}{N}\right)\left|x_k\frac{\partial \tilde{H}_P}{\partial y_k} - y_k\frac{\partial \tilde{H}_P}{\partial x_k}\right|$$

$$\leq \left(\frac{1}{N}\sum_{k=1}^{N-1}[k]_N^{2s-2}e^{2\sigma[k]_N}\omega\left(\tfrac{k}{N}\right)(y_k^2+x_k^2)\right)^{1/2}\left(\frac{1}{N}\sum_{k=1}^{N-1}[k]_N^{2s-2}e^{2\sigma[k]_N}\omega\left(\tfrac{k}{N}\right)\left(\left|\frac{\partial \tilde{H}_P}{\partial y_k}\right|^2 + \left|\frac{\partial \tilde{H}_P}{\partial x_k}\right|^2\right)\right)^{1/2}$$

$$\leq 2\|w\|_{\mathcal{P}^{s-1,\sigma}_{\mathbb{R}}}\left\|X_{\tilde{H}_P}(w)\right\|_{\mathcal{P}^{s-1,\sigma}_{\mathbb{R}}} \leq \frac{C}{N}\left[|\beta-1|\,\|w\|^4_{\mathcal{P}^{s,\sigma}} + C\|w\|^5_{\mathcal{P}^{s,\sigma}}\right] \ ,$$

where in the last inequality we used (1.125). Using that $|I_k(w(t)) - I_k(w(0))| \leq \int_0^t \left|\{I_k, \tilde{H}_P\}(w(s))\right| ds$, one gets

$$\frac{1}{N}\sum_{k=1}^{N-1}[k]_N^{2s-2}e^{2\sigma[k]_N}\omega\left(\tfrac{k}{N}\right)|I_k(w(t)) - I_k(w(0))| \leq \frac{|t|C}{N}\sup_{|s|\leq t}\left[|\beta-1|\,\|w(s)\|^4_{\mathcal{P}^{s,\sigma}} + C\|w(s)\|^5_{\mathcal{P}^{s,\sigma}}\right] \ ,$$

which, using $w(t) \in B^{s,\sigma}_{\mathbb{R}}\left(\frac{3R}{N^2}\right)$ immediately implies the thesis. □

A Properties of normally analytic maps

In this section we study the properties of the space $\mathcal{N}_\rho(\mathcal{P}^{w^1}, \mathcal{P}^{w^2})$ and $\mathcal{A}^{w^2}_{w^1,\rho}$ defined in section 2, with weights $w^1 \le w^2$. In particular, we consider the operations on germs defined in [KP10] and perform quantitative estimates.

Lemma 1.63. *Let $w^1 \le w^2 \le w^3$ be weights. Let $G \in \mathcal{N}_\rho(\mathcal{P}^{w^1}, \mathcal{P}^{w^2})$ with $|\underline{G}|_\rho \le \sigma$ and $F \in \mathcal{N}_\sigma(\mathcal{P}^{w^2}, \mathcal{P}^{w^3})$. Then $F \circ G \in \mathcal{N}_\rho(\mathcal{P}^{w^1}, \mathcal{P}^{w^3})$ and $|\underline{F \circ G}|_\rho \le |\underline{F}|_\sigma$.*

Proof. Exploiting the obvious inequality $\underline{F \circ G}(|v|) \le \underline{F} \circ \underline{G}(|v|)$(cf [KP10]), one has

$$|\underline{F \circ G}|_\rho \equiv \sup_{v \in B^{w^1}(\rho)} \|\underline{F \circ G}(|v|)\|_{w^3} \le \sup_{v \in B^{w^1}(\rho)} \|\underline{F}(\underline{G}(|v|))\|_{w^3} \le \sup_{u \in B^{w^2}(\sigma)} \|\underline{F}(|u|)\|_{w^3} \equiv |\underline{F}|_\sigma \,.$$

□

Lemma 1.64. *Let $F \in \mathcal{N}_\rho(\mathcal{P}^{w^1}, \mathcal{P}^{w^2})$, $F = O(v^2)$ and $|\underline{F}|_\rho \le \rho/e$. Then the map $\mathbb{1} + F$ is invertible in $B^{w^1}(\mu\rho)$, μ as in (1.50). Moreover there exists $G \in \mathcal{N}_{\mu\rho}(\mathcal{P}^{w^1}, \mathcal{P}^{w^2})$, $G = O(v^2)$, such that $(\mathbb{1} + F)^{-1} = \mathbb{1} - G$, and*

$$|\underline{G}|_{\mu\rho} \le \frac{|\underline{F}|_\rho}{8}. \tag{1.130}$$

Proof. We look for G in the form $G = \sum_{n \ge 2} G^n$, with the homogeneous polynomial G^n to be determined at every order n. Note that the equation defining G can be given in the form $F(v - G(v)) = G(v)$, which can be recasted in a recursive way giving the formula

$$G^n(v) = \sum_{r=2}^{n} \sum_{k_1 + \cdots + k_r = n} \tilde{F}^r\Big(G^{k_1}(v), \cdots, G^{k_r}(v)\Big), \qquad \forall\, n \ge 2 \,. \tag{1.131}$$

In the formula above $k_1, \ldots, k_r \in \mathbb{N}$, and we write $F = \sum_{r \ge 2} F^r$, where F^r is a homogeneous polynomial of degree r and $\tilde{F}^r$ is its associated multilinear map (see (1.29)). Moreover we write $G^1(v) := v$. We show now that the formal series $G = \sum_{n \ge 2} G^n$ with G^n defined by (1.131) is normally analytic in $B^{w^1}(\mu\rho)$. Note that

$$\underline{G^n}(|v|) \le \sum_{r=2}^{n} \sum_{k_1 + \cdots + k_r = n} \underline{\tilde{F}^r}\Big(\underline{G^{k_1}}(|v|), \ldots, \underline{G^{k_r}}(|v|)\Big). \tag{1.132}$$

In order to prove that the series $\sum_{n \ge 2} \underline{G^n}$ is convergent in $B^{w^1}(\mu\rho)$, we prove that there exists a constant $A > 0$ such that

$$\|\underline{G^n}(|v|)\|_{w^2} \le \frac{|\underline{F}|_\rho}{8Sn^2} A^n \|v\|^n_{w^1} \,, \qquad \forall n \ge 2. \tag{1.133}$$

The proof is by induction on n. We will use in the following the chain of inequalities

$$\left\|\underline{\tilde{F}^r}\right\| \le e^r \|\underline{F^r}\| \le e^r |\underline{F}|_\rho / \rho^r \qquad \forall r \ge 1 \,,$$

see [Muj86]. For $n=2$, by (1.131) it follows that $G^2(v)=\tilde{F}^2(v,v)$. Since

$$\left\|\underline{G^2}(|v|)\right\|_{w^2} \le \left\|\underline{\tilde{F}^2}\right\| \|v\|_{w^1}^2 \le e^2 \frac{|\underline{F}|_\rho}{\rho^2} \|v\|_{w^1}^2 ,$$

it follows that (1.133) holds for $n=2$ with $A=\frac{e(32S)^{1/2}}{\rho}$. We prove now the inductive step $n-1 \rightsquigarrow n$. Assume therefore that (1.133) holds up to order $n-1$. Then one has

$$\begin{aligned}
\|\underline{G^n}(|v|)\|_{w^2} &\le \sum_{r=2}^{n} \sum_{k_1+\cdots+k_r=n} \left\|\underline{\tilde{F}^r}\right\| \left\|\underline{G^{k_1}}(|v|)\right\|_{w^2} \cdots \left\|\underline{G^{k_r}}(|v|)\right\|_{w^2} \\
&\le A^n \|v\|_{w^1}^n \sum_{r=2}^{n} \sum_{k_1+\cdots+k_r=n} e^r \frac{|\underline{F}|_\rho}{\rho^r} \frac{|\underline{F}|_\rho^r}{8^r S^r k_1^2 \cdots k_r^2} \\
&\le \frac{|\underline{F}|_\rho}{4Sn^2} A^n \|v\|_{w^1}^n \sum_{r=2}^{\infty} \left(\frac{e\,|\underline{F}|_\rho}{2\rho}\right)^r \le \frac{|\underline{F}|_\rho}{8Sn^2} A^n \|v\|_{w^1}^n
\end{aligned}$$

where in the first inequality we used the fact that $w^1 \le w^2$, in the second the inductive assumption and in the last we used the hypothesis $|\underline{F}|_\rho \le \rho/e$. Finally to pass from the second to the third line we used the following inequality, proved in Lemma (1.67) below:

$$n^2 \sum_{k_1+\cdots+k_r=n} \frac{1}{k_1^2 \cdots k_r^2} \le (4S)^{r-1}, \qquad n \ge 1 . \tag{1.134}$$

Hence, choosing $\mu\rho = 1/A = \rho/e(32S)^{1/2}$ one proves (1.130). $\square$

Now it is easy to prove the following lemma, giving closedness of the class $\mathcal{A}_{w^1,\rho}^{w^2}$ under different operations.

Lemma 1.65. *Let $w^1 \le w^2$ be weights and let μ be as in (1.50). Then the following holds true:*

i) Let $F \in \mathcal{A}_{w^1,\rho}^{w^2}$ and $G \in \mathcal{A}_{w^1,\mu\rho}^{w^2}$ with $\|G\|_{\mathcal{A}_{w^1,\mu\rho}^{w^2}} < \frac{\mu\rho}{e}$. Then $H(v) := F(v+G(v))$ is of class $\mathcal{A}_{w^1,\mu\rho}^{w^2}$ and

$$\|H\|_{\mathcal{A}_{w^1,\mu\rho}^{w^2}} \le 2\,\|F\|_{\mathcal{A}_{w^1,\rho}^{w^2}} .$$

ii) Let $F \in \mathcal{A}_{w^1,\rho}^{w^2}$ and $\|F\|_{\mathcal{A}_{w^1,\rho}^{w^2}} \le \rho/e$. Then $(\mathbb{1}+F)^{-1} = \mathbb{1}+G$, with $G \in \mathcal{A}_{w^1,\mu\rho}^{w^2}$. Moreover one has

$$\|G\|_{\mathcal{A}_{w^1,\mu\rho}^{w^2}} \le 2\,\|F\|_{\mathcal{A}_{w^1,\rho}^{w^2}} . \tag{1.135}$$

iii) Let $F \in \mathcal{A}_{w^1,\rho}^{w^2}$, then the function $H(v) := dF(v)v$ is in the class $\mathcal{A}_{w^1,\mu\rho}^{w^2}$ and

$$\|H\|_{\mathcal{A}_{w^1,\mu\rho}^{w^2}} \le 2\,\|F\|_{\mathcal{A}_{w^1,\rho}^{w^2}} .$$

iv) Let $F^0, G^0 \in \mathcal{A}^{w^2}_{w^1,\rho}$ with $\left\|F^0\right\|_{\mathcal{A}^{w^2}_{w^1,\rho}} \le \frac{\rho}{e}$. Denote $F = \mathbb{1} + F^0$. Then $H(v) := dG^0(v)^*(F(v))$ is in the class $\mathcal{A}^{w^2}_{w^1,\mu\rho}$ and

$$\|H\|_{\mathcal{A}^{w^2}_{w^1,\mu\rho}} \le 2\left\|G^0\right\|_{\mathcal{A}^{w^2}_{w^1,\rho}} .$$

Proof. *i)* Since $\underline{H}(|v|) \le \underline{F}(|v| + \underline{G}(|v|))$ it follows that $|\underline{H}|_{\mu\rho} \le |\underline{F}|_{2\mu\rho} \le |\underline{F}|_\rho$. Furthermore, since $dH(v) = dF(v + G(v))(\mathbb{1} + dG(v))$ one gets that $\underline{dH}(|v|) \le \underline{dF}(|v| + \underline{G}(|v|)) + \underline{dF}(|v| + \underline{G}(|v|))\underline{dG}(|v|)$, which implies that $\mu\rho\,|\underline{dH}|_{\mu\rho} \le |\underline{dF}|_\rho\,(\mu\rho + \mu\rho\,|\underline{dG}|_{\mu\rho}) \le |\underline{dF}|_\rho\,\mu\rho(1 + 1/e)$. The adjoint $dH(v)^*$ is estimated analogously, thus the claimed estimate follows.

ii) It follows from the formula $dG(v) = [\mathbb{1} - dF(v - G(v))]^{-1} dF(v - G(v))$, arguing as in item *i)*.

iii) It follows from $dH(v)u = dF(v)u + d^2F(v)(u, v)$, arguing as in item *i)*.

iv) To estimate $\underline{H}(|v|)$ and $\underline{dH}(|v|)$ one proceeds as in item *i)*. In order to estimate $\underline{dH}(|v|)^*$ remark that (see [KP10]) $dH(v)^*u = (dF^0(v)^* + \mathbb{1})dG^0(v)u + d_v(dG^0(v)^*u)(F(v))$, thus

$$\underline{dH}(|v|)^*|u| \le (\underline{dF^0}(|v|)^* + \mathbb{1})\underline{dG^0}(|v|)|u| + d_{|v|}(\underline{dG^0}(|v|)^*|u|)(\underline{F}(|v|)) .$$

The claimed estimate follows easily.

□

Now we analyze the flow generated by a vector field of class $\mathcal{A}^{w^2}_{w^1,\rho}$. Given a time dependent vector field $V_t(v)$, consider the differential equation

$$\begin{cases} \dot{u}(t) = V_t(u(t)) \\ u(0) = v . \end{cases} \tag{1.136}$$

We will denote by $\phi^t(v)$ the corresponding flow map whose existence and properties are given in the next lemma.

Lemma 1.66. *Assume that the map $[0,1] \ni t \mapsto V_t \in \mathcal{A}^{w^2}_{w^1,\rho}$ is continuous and furthermore fulfills $\sup_{t\in[0,1]} \|V_t\|_{\mathcal{A}^{w^2}_{w^1,\rho}} \le \rho/e$; then for each $t \in [0,1]$, $\phi^t - \mathbb{1} \in \mathcal{A}^{w^2}_{w^1,\mu\rho}$ with μ as in* (1.50)*. Furthermore one has*

$$\left\|\phi^t - \mathbb{1}\right\|_{\mathcal{A}^{w^2}_{w^1,\mu\rho}} \le 2 \sup_{t\in[0,1]} \|V_t\|_{\mathcal{A}^{w^2}_{w^1,\rho}} . \tag{1.137}$$

Proof. We look for a solution $u(t,v) = \sum_{j\ge1} u^j(t,v)$ in power series of v, with $u^j(t,v)$ a homogeneous polynomial of degree j in v. Expanding the vector field $V_t(v) = \sum_{r\ge2} V^r_t(v)$ in Taylor series, one obtains the recursive formula for the solution

$$u^1(t,v) = v, \qquad u^n(t,v) = \sum_{r=2}^{n} \sum_{k_1+\cdots+k_r=n} \int_0^t \tilde{V}^r_s(u^{k_1}(s,v), \ldots, u^{k_r}(s,v))\, ds \qquad \forall n \ge 2, \tag{1.138}$$

where $\tilde{V}^r_s$ is the multilinear map associated to V^r_s (see (1.29)). Arguing as in the proof of (1.64) one gets the bounds

$$\|\underline{u}^n(t,v)\|_{w^2} \le \frac{\sup_{t\in[0,1]} |\underline{V_t}|_\rho}{8Sn^2} A^n \|v\|^n_{w^1} \qquad \forall n \ge 2, \tag{1.139}$$

with $A = \frac{e}{\rho}(32S)^{1/2}$, from which it follows that $\left|\underline{\phi^t - \mathbb{1}}\right|_{\mu\rho} \leq \sup_{t\in[0,1]} \left|\underline{V_t}\right|_\rho /8$.

We come to the estimate of the differential of $u(t,v)$ and of its adjoint. We differentiate equation (1.138) getting the recursive formula

$$du^n(t,v)\xi = \sum_{r=2}^{n} \sum_{k_1+\cdots+k_r=n} \int_0^t \left[\tilde{V}_s^r(du^{k_1}(s,v)\xi, \ldots, u^{k_r}(s,v)) + \cdots + \tilde{V}_s^r(u^{k_1}(s,v), \ldots, du^{k_r}(s,v)\xi)\right] ds \,. \tag{1.140}$$

To estimate such an expression remark that, defining $E_t(v) := dV_t(v)$ (where the differential is with respect to the v variable only), one has

$$d^{r-1}E_s(u^{k_2}(s,v), \ldots, u^{k_r}(s,v))\xi = \tilde{V}_s^r(\xi, u^{k_2}(s,v), \ldots, u^{k_r}(s,v))$$

which allows to write formula (1.140) as

$$\begin{aligned} du^n(t,v)\xi = \sum_{r=2}^{n} \sum_{k_1+\cdots+k_r=n} \int_0^t &\left[d^{r-1}E_s(u^{k_2}(s,v) \ldots, u^{k_r}(s,v))du^{k_1}(s,v)\xi + \ldots\right. \\ &\left.\ldots + d^{r-1}E_s(u^{k_1}(s,v), \ldots, u^{k_{r-1}}(s,v))du^{k_r}(s,v)\xi\right] ds \,. \end{aligned} \tag{1.141}$$

This formula allows to proceed exactly as in the estimate of $\underline{u^n}$, namely making the inductive assumption that

$$\|\underline{du^n}(t,v)\|_{\mathcal{L}(\mathcal{P}^{w^1},\mathcal{P}^{w^2})} \leq \frac{\sup_{t\in[0,1]} \left|\underline{dV_t}\right|_\rho}{8Sn^2} A^n \|v\|_{w^1}^n$$

and proceeding as above one gets the thesis. Finally one has to estimate $[\underline{du^n}]^*$, but again equation (1.141) allows to obtain a formula whose estimate is obtained exactly as the estimate of $\underline{du}$. □

We prove now a useful inequality.

Lemma 1.67. *[Trè70] Let $r \in \mathbb{N}$ be fixed and $S = \sum_{k\geq 1} \frac{1}{k^2}$. Then for every $n \in \mathbb{N}$ it holds that*

$$n^2 \sum_{\substack{k_1,\ldots,k_r\in\mathbb{N} \\ k_1+\cdots+k_r=n}} \frac{1}{k_1^2 \cdots k_r^2} \leq (4S)^{r-1} \,.$$

Proof. The proof is by induction, the case $n = 1$ being trivial. For $n > 1$ one gets

$$n^2 \sum_{k_1+\cdots+k_r=n} \frac{1}{k_1^2 \cdots k_r^2} = \sum_{k_1+j=n} \frac{n^2}{k_1^2 j^2} \sum_{k_2+\cdots+k_r=j} \frac{j^2}{k_2^2 \cdots k_r^2} \leq \sum_{k_1+j=k} \frac{n^2}{k_1^2 j^2}(4S)^{r-2}$$

by the induction assumption. Now it is enough to note that

$$\sum_{k_1+j=n} \frac{n^2}{k_1^2 j^2} = \sum_{k_1+j=n} \frac{n^2}{k_1^2(n-k_1)^2} \leq 2\sum_{k_1=1}^{n-1} \left(\frac{1}{k_1^2} + \frac{1}{(n-k_1)^2}\right) \leq 4\sum_{k_1=1}^{n-1} \frac{1}{k_1^2} \leq 4S.$$

□

B Discrete Fourier Transform

In this section we collect some well-known properties of the discrete Fourier transform (DFT). For $u \in \mathbb{C}^N$, $N \in \mathbb{N}$, the DFT of u is the vector $\hat{u} \in \mathbb{C}^N$ whose k^{th} component is defined by

$$\hat{u}_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} u_j e^{2\pi \mathrm{i} jk/N}, \qquad \forall\, 0 \le k \le N-1. \tag{1.142}$$

When the DFT is considered as a map, it will be denoted by $\mathcal{F}$, i.e. $\mathcal{F} : u \mapsto \hat{u}$.

For any $s \ge 0$ and $\sigma \ge 0$ we endow $\mathbb{C}^N$ with the norm $\|\cdot\|_{s,\sigma}$ defined in (1.113). Such a space will be denoted by $\mathbb{C}^{s,\sigma}$.

Remark 1.68. *Let j be an integer such that $0 \le j \le N-1$. Then*

$$\sum_{k=0}^{N-1} e^{\mathrm{i}2\pi jk/N} = \begin{cases} 0 & \text{if } j \neq 0 \\ N & \text{if } j = 0 \end{cases} \qquad \textit{and} \qquad \sum_{k=0}^{2N-1} u_k\, e^{\mathrm{i}\pi kj/N} = \begin{cases} 2\sqrt{N}\,\hat{u}_l, & j \text{ even, } j = 2l \\ 0 & j \text{ odd} \end{cases} \tag{1.143}$$

Remark 1.69. *Fix $s > \frac{1}{2}$ and $\sigma \ge 0$. Then there exists a constant $C_{s,\sigma} > 0$, independent of N, such that for every $u \in \mathbb{C}^N$ the following estimate holds:*

$$\sup_{0 \le j \le N-1} |u_j| \le C_{s,\sigma} \|u\|_{s,\sigma} \ .$$

For $u, v \in \mathbb{C}^N$, we denote by $u \cdot v$ the component-wise product of u and v, namely the vector whose j^{th} component is given by the product of the j^{th} components of u and v:

$$(u \cdot v)_j := u_j v_j, \qquad 0 \le j \le N-1 \ . \tag{1.144}$$

We denote by $u * v$ the convolution product of u and v, a vector whose j^{th} component is defined by

$$(u * v)_j := \sum_{k=0}^{N-1} u_k v_{j-k}, \qquad 0 \le j \le N-1 \ , \tag{1.145}$$

where in the summation above u and v are extended periodically defining $v_{k+lN} \equiv v_k$ for $l \in \mathbb{Z}$. The DFT maps the component-wise product in convolution:

Lemma 1.70. *For $s > \frac{1}{2}$ and $\sigma \ge 0$ there exists a constant $C_{s,\sigma} > 0$, independent of N, such that the following holds:*

(i) $\widehat{u \cdot v} = \frac{1}{\sqrt{N}}\, \hat{u} * \hat{v}$;

(ii) $\|u \cdot v\|_{s,\sigma} \le C_{s,\sigma} \|u\|_{s,\sigma} \|v\|_{s,\sigma}$;

(iii) the map $X : u \mapsto u^2$, has bounded modulus w.r.t. the exponentials, and $\|\underline{X}(u)\|_{s,\sigma} \le C_{s,\sigma} \|u\|_{s,\sigma}^2$.

Proof. Item (i) is standard and the details of the proof are omitted.

We prove now item (ii). To begin, note that, by periodicity, one has

$$\|u\|_{s,\sigma}^2 = \frac{1}{N}\sum_{k\in K_N^0} [k]^{2s} e^{2\sigma|k|}\,|\hat{u}_k|^2 \ ,$$

where the set

$$K_N^0 := \{k \in \mathbb{Z} : -(N-1)/2 \le k \le (N-1)/2\} \cup \{\lfloor N/2 \rfloor\}, \tag{1.146}$$

while $[k] := \max(1, |k \bmod N|)$. By item (i), one has that

$$\|u\cdot v\|_{s,\sigma}^2 = \frac{1}{N}\sum_{k\in K_N^0} [k]^{2s} e^{2\sigma|k|} |\widehat{(u\cdot v)}_k|^2 = \frac{1}{N^2}\sum_{k\in K_N^0} [k]^{2s} e^{2\sigma|k|} \left|\sum_{l=0}^{N-1} \hat{u}_l \hat{v}_{k-l}\right|^2 . \tag{1.147}$$

Introduce now the quantities

$$\gamma_{k,l} := \frac{[k]^s}{[l]^s\,[k-l]^s}\cdot\frac{e^{\sigma|k|}}{e^{\sigma|l|}e^{\sigma|k-l|}} \ .$$

For $s > \frac{1}{2}$ and $\sigma \ge 0$, it holds that $\gamma_{k,l}^2 \le 4^s \frac{([k-l]^{2s}+[l]^{2s})\,e^{2\sigma(|k-l|+|l|)}}{[k-l]^{2s}\,[l]^{2s}\,e^{2\sigma|l|}\,e^{2\sigma|k-l|}} \le 4^s\left(\frac{1}{[l]^{2s}} + \frac{1}{[k-l]^{2s}}\right)$, from which it follows that there exists a constant $C_{s,\sigma} > 0$, independent of N, such that

$$\sup_{0\le k\le N-1} \sum_{l=0}^{N-1} \gamma_{k,l}^2 \le C_{s,\sigma}^2 \ . \tag{1.148}$$

By Cauchy-Schwartz one has

$$\begin{aligned}
[k]^s e^{\sigma|k|} \sum_{l=0}^{N-1} |\hat{u}_l|\,|\hat{v}_{k-l}| &= \sum_{l=0}^{N-1} \gamma_{k,l}\,[l]^s\,e^{\sigma|l|}\,|\hat{u}_l|\,[k-l]^s\,e^{\sigma|k-l|}\,|\hat{v}_{k-l}| \\
&\le \left(\sum_{l=0}^{N-1}\gamma_{k,l}^2\right)^{1/2} \left(\sum_{l=0}^{N-1} [l]^{2s}\,e^{2\sigma|l|}\,|\hat{u}_l|^2\,[k-l]^{2s}\,e^{2\sigma|k-l|}\,|\hat{v}_{k-l}|^2\right)^{1/2} .
\end{aligned}$$

Inserting the inequality above in (1.147), one has

$$\begin{aligned}
\|u\cdot v\|_{s,\sigma} &\le \frac{C_{s,\sigma}}{N}\left(\sum_{l=0}^{N-1}[l]^{2s}\,e^{2\sigma|l|}\,|\hat{u}_l|^2\right)^{1/2} \left(\sum_{k=0}^{N-1}[k-l]^{2s}\,e^{2\sigma|k-l|}\,|\hat{v}_{k-l}|^2\right)^{1/2} \\
&\le C_{s,\sigma}\,\|u\|_{s,\sigma}\,\|v\|_{s,\sigma} .
\end{aligned}$$

We prove now item (iii). Consider $\widehat{X} := \mathcal{F}X\mathcal{F}^{-1}$. By item (i) one has $\widehat{X} : \{\hat{u}_j\}_{j\in\mathbb{Z}} \mapsto \{\frac{1}{\sqrt{N}}\sum_l \hat{u}_l\hat{u}_{j-l}\}_{j\in\mathbb{Z}}$. Thus $\underline{\widehat{X}} \equiv \widehat{X}$ and the claim follows. □

Remark 1.71. *Let* $S_\pm$ *be the difference operators defined in* (1.117). *Let* $\hat{\omega}_\pm$ *be the vectors whose* k^{th} *components are given by* $\hat{\omega}_{\pm,k} := 1 - e^{\mp 2\pi \mathrm{i} k/N}$. *Then the following holds:*

(i) the map $\widehat{S}_\pm := \mathcal{F} S_\pm \mathcal{F}^{-1}$ *is a multiplication by the vector* $\hat{\omega}_\pm$*:* $\widehat{S}_\pm : \hat{u} \mapsto \hat{\omega}_\pm \cdot \hat{u}$.

(ii) $\left|\underline{\widehat{S}_\pm}(\hat{u})\right| \leq \omega \cdot |\hat{u}|$*, where* $\omega \equiv \{\omega\left(\frac{k}{N}\right)\}_{k=1}^{N-1}$ *is the vector of the linear frequencies.*

Remark 1.72. *Consider* $q = q(\xi, \eta)$ *as a function of the linear Birkhoff variables defined in* (1.25). *Then one has* $\left\|\underline{S_\pm}(q)\right\|_{s,\sigma} \leq \|(\xi, \eta)\|_{\mathcal{P}^{s,\sigma}}$.

C Proof of Proposition 1.47

We prove now property (Θ1). Let $T : (\xi, \eta) \mapsto (p, q)$ be the map introducing linear Birkhoff coordinated. Explicitly $(p, q) = T(\xi, \eta)$ iff $(\hat{p}_0,\, \hat{q}_0) = (0, 0)$ and

$$(\hat{p}_k,\, \hat{q}_k) = \left(\sqrt{\frac{1}{2}\omega\left(\tfrac{k}{N}\right)}\,(\xi_k + \eta_{N-k}),\ \frac{1}{\mathrm{i}\sqrt{2\omega\left(\frac{k}{N}\right)}}(\xi_k - \eta_{N-k})\right),\ 1 \leq k \leq N-1\ .$$

Then $\Theta_\Xi \equiv \Theta \circ T$ and in particular $d\Theta_\Xi(0,0) = d\Theta(0,0)T$. Using the formula above and the fact that $d\Theta(0,0)(P, Q) = \left(-P, \frac{1}{2}S_+(Q)\right)$, where S_+ is defined in (1.117), one obtains easily formula (1.86). The estimate of $\left\|\underline{d\Theta_\Xi}(0,0)\right\|_{\mathcal{L}(\mathcal{P}^{s,\sigma},\, \mathcal{C}^{s,\sigma})}$ is trivial, and is omitted.

We prove now the estimate for $\left\|\underline{d\Theta_\Xi}(0,0)^*\right\|_{\mathcal{L}(\mathcal{C}^{s+1,\sigma},\, \mathcal{P}^{s,\sigma})}$. Using the explicit formula (1.86), one computes that $(\xi, \eta) = d\Theta_\Xi(0,0)^*(B, A)$ iff

$$(\xi_k,\ \eta_k) = \left(-\sqrt{\tfrac{1}{2}\omega\left(\tfrac{k}{N}\right)}\hat{B}_k + \frac{\varpi_k}{\mathrm{i}\sqrt{2\omega\left(\frac{k}{N}\right)}}\hat{A}_k,\ -\sqrt{\tfrac{1}{2}\omega\left(\tfrac{k}{N}\right)}\hat{B}_{N-k} - \frac{\varpi_k}{\mathrm{i}\sqrt{2\omega\left(\frac{k}{N}\right)}}\hat{A}_{N-k}\right)$$

for $1 \leq k \leq N-1$. Thus there exist constants $C, C_{\Theta_2} > 0$, independent of N, such that

$$\left\|\underline{d\Theta_\Xi}(0,0)^*(B, A)\right\|_{\mathcal{P}^{s,\sigma}} \leq C\left(\frac{1}{N}\sum_{k=1}^{N-1}[k]_N^{2s}e^{2\sigma[k]_N}\omega\left(\tfrac{k}{N}\right)^2\left(|\widehat{B}_k|^2 + |\widehat{A}_k|^2\right)\right)^{1/2} \leq \frac{C_{\Theta_2}}{N}\|(B, A)\|_{\mathcal{C}^{s+1,\sigma}}\ ,$$

where we used that $\left|\omega\left(\frac{k}{N}\right)\right|^2 \leq \frac{\pi^2[k]_N^2}{N^2}$. Thus the second of (1.87) is proved.

We prove now property (Θ2). Denote by Θ_b the map $p \mapsto -p$ and by Θ_a the map $q \mapsto \exp\left(\frac{1}{2}S_+(q)\right) - 1$. Then $(b, a) = \Theta(p, q) \equiv (\Theta_b(p), \Theta_a(q))$. Introduce on $\mathbb{C}^N$ the norm $\|\cdot\|_{s,\sigma}$ defined in (1.113). Then $\|\Theta(p, q)\|^2_{\mathcal{C}^{s,\sigma}} \equiv \|\Theta_b(p)\|^2_{s,\sigma} + \|\Theta_a(q)\|^2_{s,\sigma}$. The analyticity of $p \mapsto \Theta_b(p)$ is obvious. Consider now the map $q \mapsto \Theta_a(q)$. Expand Θ_a in Taylor series with center at the origin to get

$$\Theta_a(q) = \sum_{r\geq 1}\Theta_a^r(q), \qquad \Theta_a^r(q) := \frac{1}{r!\,2^r}(S_+(q))^r, \qquad \forall r \geq 1. \tag{1.149}$$

Consider q as a function of the linear Birkhoff variables ξ, η. Then Lemma 1.70 and Remark 1.72 imply that for any $s \geq 0$, $\sigma \geq 0$

$$\left\|\underline{\Theta_a^r}(q)\right\|_{s+1,\sigma} \leq C_1^r\left\|\underline{S_+}(q)\right\|^r_{s+1,\sigma} \leq C_2^r\|(\xi, \eta)\|^r_{\mathcal{P}^{s+1,\sigma}} \leq C_3^r N^r\|(\xi, \eta)\|^r_{\mathcal{P}^{s,\sigma}}\ ,\ \forall r \geq 2, \tag{1.150}$$

where $C_1, C_2, C_3 > 0$ are positive constants independent of N. Therefore for $\epsilon < \frac{1}{C_3}$ one has

$$\sup_{\|(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}} \leq \epsilon/N^2} \left\| \underline{\Theta^0_{\Xi}}(\xi,\eta) \right\|_{\mathcal{C}^{s+1,\sigma}} \leq \sum_{r\geq 2} \sup_{\|(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}} \leq \epsilon/N^2} \left\| \underline{\Theta^r_{\Xi}}(\xi,\eta) \right\|_{\mathcal{C}^{s+1,\sigma}} \leq \sum_{r\geq 2} C_3^r N^r \frac{\epsilon^r}{N^{2r}} \leq \frac{2C_3^2 \epsilon^2}{N^2} \ .$$

This proves the first estimate in (Θ2). We show now that for any $s \geq 0$, $\sigma \geq 0$ one has $[d\Theta^0_{\Xi}]^* \in \mathcal{N}_{\epsilon/N^2}(\mathcal{P}^{s,\sigma}, \mathcal{L}(\mathcal{C}^{s+2,\sigma}, \mathcal{P}^{s+1,\sigma}))$. Note that $d\Theta_{\Xi}(\xi,\eta)^* = T^* d\Theta(T(\xi,\eta))^*$. Using the explicit expression of T, one verifies that $(\xi,\eta) = T^*(P,Q)$ iff

$$(\xi_k,\, \eta_k) = \left(\sqrt{\tfrac{1}{2}\omega\left(\tfrac{k}{N}\right)} \widehat{P}_k + \frac{1}{\mathrm{i}\sqrt{2\omega\left(\frac{k}{N}\right)}} \widehat{Q}_k,\ \sqrt{\tfrac{1}{2}\omega\left(\tfrac{k}{N}\right)} \widehat{P}_{N-k} - \frac{1}{\mathrm{i}\sqrt{2\omega\left(\frac{k}{N}\right)}} \widehat{Q}_{N-k}, \right) \tag{1.151}$$

for $1 \leq k \leq N-1$. Thus one has that for any $s \geq 0$, $\sigma \geq 0$

$$\|T^*(0,Q)\|_{\mathcal{P}^{s,\sigma}} \leq \|Q\|_{s,\sigma} \ . \tag{1.152}$$

Using (1.149) one verifies that $d\Theta^r(p,q)(P,Q) = \frac{1}{(r-1)!\,2^r}\left(0,\, S_+(q)^{r-1} \cdot S_+(Q)\right)$, $\forall r \geq 2$, from which it follows that

$$d\Theta^r(p,q)^*(B,A) = \frac{1}{(r-1)!\,2^r}\left(0,\ \overline{S_+(q)}^{\,r-1} \cdot S_-(A)\right) \ , \qquad \forall r \geq 2 \ .$$

Thus, using estimate (1.152), there exists a constant $C_4 > 0$, independent of N, such that

$$\left\| \underline{d\Theta^r_{\Xi}}(\xi,\eta)^*(B,A) \right\|_{\mathcal{P}^{s+1,\sigma}} \leq C_4^r \left\| \underline{S_+}(\underline{q}(\xi,\eta)) \right\|^{r-1}_{s+1,\sigma} \left\| \underline{S_-}(A) \right\|_{s+1,\sigma} \leq C_4^r N^{r-2} \|(\xi,\eta)\|^{r-1}_{\mathcal{P}^{s,\sigma}} \|(B,A)\|_{\mathcal{C}^{s+2,\sigma}} \ .$$

Then there exists $C_5, \epsilon_0 > 0$, independent of N, such that $\forall\, 0 < \epsilon \leq \epsilon_0$

$$\begin{aligned}\sup_{\|(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}} \leq \epsilon/N^2} \left\| \underline{d\Theta^0_{\Xi}}(\xi,\eta)^* \right\|_{\mathcal{L}(\mathcal{C}^{s+2,\sigma},\, \mathcal{P}^{s+1,\sigma})} &\leq \sum_{r\geq 2} \sup_{\|(\xi,\eta)\|_{\mathcal{P}^{s,\sigma}} \leq \epsilon/N^2} \left\| \underline{d\Theta^r_{\Xi}}(\xi,\eta)^* \right\|_{\mathcal{L}(\mathcal{C}^{s+2,\sigma},\, \mathcal{P}^{s+1,\sigma})} \\ &\leq \sum_{r\geq 2} C_4^r N^{r-2} \frac{\epsilon^{r-1}}{N^{2(r-1)}} \leq \frac{C_5 \epsilon}{N^2}.\end{aligned}$$

D Proof of Lemma 1.50 and Corollary 1.51

Proof of Lemma 1.50. Since the map $(b,a) \mapsto L_p(b,a)$ is linear, it is enough to prove that it is continuous from $\mathcal{C}^{s,\sigma}$ to $\mathcal{L}(\mathbb{C}^{2N})$. In particular we will prove that

$$\|L_p\|_{\mathcal{L}(\mathbb{C}^{2N})} \leq \sup_{0\leq j\leq N-1} \left(|b_j| + 2\sup_j |a_j| \right). \tag{1.153}$$

This estimate, together with Lemma 1.69, proves (1.93). In order to prove (1.153), write $L_p = D + A^+ + A^-$, where D is the diagonal part of L_p and $A^\pm$ are defined by

$$A^+ = \begin{pmatrix} 0 & a_0 & & & \\ & 0 & \ddots & & \\ & & 0 & a_{N-1} & \\ a_{N-1} & & & 0 & \end{pmatrix}, \quad A^- = \begin{pmatrix} 0 & & & a_{N-1} \\ a_0 & 0 & & \\ & \ddots & 0 & \\ & & a_{N-2} & 0 \end{pmatrix}.$$

To estimate the norms of D, A^+ and A^- is enough to observe that for every $x \in \mathbb{C}^{2N}$ one has

$$\|Dx\|_{\mathbb{C}^{2N}}^2 := \sum_{j=0}^{2N-1} |b_j x_j|^2 \leq \left(\sup_{0 \leq j \leq N-1} |b_j| \right)^2 \|x\|_{\mathbb{C}^{2N}}^2 , \qquad \left\|A^{\pm} x\right\|_{\mathbb{C}^{2N}}^2 \leq \left(\sup_{0 \leq j \leq N-1} |a_j| \right)^2 \|x\|_{\mathbb{C}^{2N}}^2 ,$$

where $\|\cdot\|_{\mathbb{C}^{2N}}$ is the standard euclidean norm on $\mathbb{C}^{2N}$. Thus (1.153) follows. □

Proof of Corollary 1.51. Item (*i*) follows by standard perturbation theory, and the details are omitted. We prove now item (*ii*). Let Γ_j be the circle defined by $\Gamma_j := \left\{ \lambda \in \mathbb{C} : \left|\lambda_{2j}^0 - \lambda\right| = \frac{1}{2N^2} \right\}$, counter-clockwise oriented. By item (*i*), for any $\|(b,a)\|_{\mathcal{C}^{s,\sigma}} \leq \frac{\epsilon_*}{N}$, $\lambda_{2j}(b,a)$ and $\lambda_{2j-1}(b,a)$ are inside the ball enclosed by Γ_j. Write $L_{b,a} - \lambda = L_0 - \lambda + L_p = (L_0 - \lambda)\left(1 + (L_0 - \lambda)^{-1} L_p\right)$; its inverse

$$(L_{b,a} - \lambda)^{-1} = \left(\sum_{n=0}^{\infty} \left(-(L_0 - \lambda)^{-1} L_p \right)^n \right) (L_0 - \lambda)^{-1} \tag{1.154}$$

is well defined as a Neumann operator when $\left\|(L_0 - \lambda)^{-1} L_p\right\|_{\mathcal{L}(\mathbb{C}^{2N})} < 1$. Since $L_0 - \lambda$ is diagonalizable with $\{(\lambda_j^0 - \lambda)\}_{0 \leq j \leq 2N-1}$ as eigenvalues, the norm of its inverse is bounded by the inverse of the smallest eigenvalue:

$$\sup_{\lambda \in \Gamma_j} \left\|(L_0 - \lambda)^{-1}\right\|_{\mathcal{L}(\mathbb{C}^{2N})} \leq \sup_{\substack{\lambda \in \Gamma_j \\ 0 \leq k \leq 2N-1}} \left| \frac{1}{\lambda_k^0 - \lambda} \right| < 2N^2 \tag{1.155}$$

where the last estimates is due to the form of Γ_j. Therefore for $0 < \epsilon \leq \epsilon_*$ and $\|(b,a)\|_{\mathcal{C}^{s,\sigma}} < \frac{\epsilon}{N^2}$ one gets, using (1.93),

$$\left\|(L_0 - \lambda)^{-1} L_p\right\|_{\mathcal{L}(\mathbb{C}^{2N})} \leq \|L_p\|_{\mathcal{L}(\mathbb{C}^{2N})} \left\|(L_0 - \lambda)^{-1}\right\|_{\mathcal{L}(\mathbb{C}^{2N})} \leq C_{s,\sigma} \|(b,a)\|_{\mathcal{C}^{s,\sigma}} 2N^2 < 2C_{s,\sigma}\epsilon_*,$$

which proves the convergence of the Neumann series (1.154) for $\epsilon_* \leq \frac{1}{2C_{s,\sigma}}$.
Substituting (1.154) in (1.91) we get, for $1 \leq j \leq N-1$,

$$P_j(b,a) = P_{j0} - \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_j} \left(\sum_{n=1}^{\infty} \left(-(L_0 - \lambda)^{-1} L_p \right)^n \right) (L_0 - \lambda)^{-1} \, \mathrm{d}\lambda. \tag{1.156}$$

Since the series inside the integral is absolutely and uniformly convergent for $(b,a) \in B^{\mathcal{C}^{s,\sigma}}\left(\frac{\epsilon}{N^2}\right)$, $(b,a) \mapsto P_j(b,a)$ is analytic as a map from $B^{\mathcal{C}^{s,\sigma}}\left(\frac{\epsilon}{N^2}\right)$ to $\mathcal{L}(\mathbb{C}^{2N})$. Estimate (1.95) follows easily from (1.156).

We prove now item (*iii*). Properties $(U1) - (U3)$ are standard [Kat66]. The analyticity of the map $(b,a) \mapsto U_j(b,a)$ follows from item (*ii*). Indeed, in order for $U_j(b,a)$ to be defined as a Neumann series one needs $\|P_j(b,a) - P_{j0}\|_{\mathcal{L}(\mathbb{C}^{2N})} < 1$, which follows from (1.95). Estimate (1.96) follows by expanding (1.92) in power series of $P_j(b,a) - P_{j0}$. □

E Proof of Proposition 1.52

Denote by $D : \mathbb{C}^{N-1} \to \mathbb{C}^{N-1}$ the diagonal operator

$$D : \{\xi_j\}_{1 \leq j \leq N-1} \mapsto \{D_j \xi_j\}_{1 \leq j \leq N-1}, \quad \text{where} \quad D_j := \left(\tfrac{2}{N}\omega\left(\tfrac{j}{N}\right)\right)^{-1/2} . \tag{1.157}$$

Proof of properties $(Z1)-(Z3)$. Property $(Z1)$ follows from formula (1.98), since [4]:

$$\begin{aligned}\overline{z_j(b,a)} &= D_j\overline{\left\langle\left(L_{b,a}-\lambda_{2j}^0\right)U_jf_{2j,0},\overline{U_jf_{2j,0}}\right\rangle} = D_j\left\langle\overline{U_jf_{2j,0}},\left(L_{b,a}-\lambda_{2j}^0\right)U_jf_{2j,0}\right\rangle = \\ &= D_j\left\langle U_jf_{2j-1,0},\left(L_{b,a}-\lambda_{2j}^0\right)\overline{U_jf_{2j-1,0}}\right\rangle = D_j\left\langle\left(L_{b,a}-\lambda_{2j}^0\right)f_{2j-1},\overline{f_{2j-1}}\right\rangle = w_j(b,a).\end{aligned}$$

We prove now $(Z2)$. Using Lemma 1.51 (iv) and the fact that $\overline{f_{2j,0}}=f_{2j-1,0}$, decompose $f_{2j,0}$ and f_{2j} in real and imaginary part:

$$\begin{aligned}f_{2j,0} &= e_{j,0}+\mathrm{i}h_{j,0}, \quad f_{2j}=e_j+\mathrm{i}h_j\\ f_{2j-1,0} &= e_{j,0}-\mathrm{i}h_{j,0}, \quad f_{2j-1}=e_j-\mathrm{i}h_j,\end{aligned}$$

where

$$e_{j,0}:=Re\,f_{2j,0}, \quad h_{j,0}:=Im\,f_{2j,0}, \qquad \text{and} \qquad e_j:=Re\,f_{2j}=U_je_{j,0}, \quad h_j:=Im\,f_{2j}=U_jh_{j,0}.$$

The vectors $\{e_j,h_j\}$ form a real orthogonal basis for $E_j(b,a)$. Let $M_j(b,a)$ be the matrix of the selfadjoint operator $L_{b,a}-\lambda_{2j}^0\big|_{E_j(b,a)}$ with respect to this basis:

$$M_j(b,a)=\begin{pmatrix}\alpha_j & \sigma_j\\ \sigma_j & \beta_j\end{pmatrix}.$$

The eigenvalues of M_j are obviously $\lambda_{2j}-\lambda_{2j}^0$ and $\lambda_{2j-1}-\lambda_{2j}^0$, hence

$$\begin{aligned}\text{Tr }M_j &= \alpha_j+\beta_j=\left(\lambda_{2j}-\lambda_{2j}^0\right)+\left(\lambda_{2j-1}-\lambda_{2j}^0\right),\\ \text{Det }M_j &= \alpha_j\beta_j-\sigma_j^2=\left(\lambda_{2j}-\lambda_{2j}^0\right)\left(\lambda_{2j-1}-\lambda_{2j}^0\right).\end{aligned}$$

Now observe that

$$\begin{aligned}z_j(b,a) &= D_j\left\langle\left(L_{b,a}-\lambda_{2j}^0\right)(e_j+\mathrm{i}h_j),(e_j-\mathrm{i}h_j)\right\rangle = \\ &= D_j\left\langle\left(L_{b,a}-\lambda_{2j}^0\right)e_j,e_j\right\rangle-D_j\left\langle\left(L_{b,a}-\lambda_{2j}^0\right)h_j,h_j\right\rangle+2\mathrm{i}D_j\left\langle\left(L_{b,a}-\lambda_{2j}^0\right)e_j,h_j\right\rangle = \\ &= \left(\tfrac{2}{N}\omega\left(\tfrac{j}{N}\right)\right)^{-1/2}(\alpha_j-\beta_j+\mathrm{i}2\sigma_j).\end{aligned}$$

Finally one computes

$$\begin{aligned}\left(\lambda_{2j}-\lambda_{2j-1}\right)^2 &= (\text{Tr }M_j)^2-4\text{Det }M_j=\left(\alpha_j+\beta_j\right)^2-4\alpha_j\beta_j+4\sigma_j^2\\ &= \left(\alpha_j-\beta_j\right)^2+4\sigma_j^2=\left(\text{Re }z_j\right)^2+\left(\text{Im }z_j\right)^2=\left(\tfrac{2}{N}\omega\left(\tfrac{j}{N}\right)\right)|z_j(b,a)|^2.\end{aligned}$$

We prove now $(Z3)$. The first order terms of z_j and w_j in $(b,\ a)$ are given by

$$dz_j(0,0)(b,a)=D_j\left\langle L_pf_{2j,0},\overline{f_{2j,0}}\right\rangle, \quad dw_j(0,0)(b,a)=D_j\left\langle L_pf_{2j-1,0},\overline{f_{2j-1,0}}\right\rangle, \qquad 1\le j\le N-1\,.$$

Using the explicit formula for $f_{2j,0}$ in Lemma 1.48, one computes

$$\begin{aligned}\left\langle L_pf_{2j,0},\overline{f_{2j,0}}\right\rangle &= \frac{1}{2N}\sum_{l=0}^{2N-1}b_le^{\mathrm{i}2\rho_jl}+a_{l-1}e^{\mathrm{i}2\rho_j(l-1)}e^{\mathrm{i}\rho_j}+a_le^{\mathrm{i}2\rho_jl}e^{\mathrm{i}\rho_j}\\ &= \frac{1}{2N}\sum_{l=0}^{2N-1}b_le^{\mathrm{i}2\pi jl/N}+a_{l-1}e^{\mathrm{i}2\pi(l-1)j/N}e^{\mathrm{i}\rho_j}+a_le^{\mathrm{i}2\pi lj/N}e^{\mathrm{i}\rho_j} \qquad (1.158)\\ &= \frac{1}{\sqrt{N}}\left(\hat{b}_j+2e^{\mathrm{i}\rho_j}\hat{a}_j\right)=\frac{1}{\sqrt{N}}\left(\hat{b}_j-2e^{\mathrm{i}\pi j/N}\hat{a}_j\right).\end{aligned}$$

[4] to simplify the notation, we write $f_j\equiv f_j(b,a)$ and $U_j\equiv U_j(b,a)$

The formula for $dz_j(0,0)(b,a)$ immediately follows. The one for $dw_j(0,0)(b,a)$ is proved in the same way and the details are omitted.

The estimate (1.101) for $\underline{dZ}(0,0)$ follows immediately. We estimate now the norm of $\underline{dZ}(0,0)^*$. One checks that $(B,A) = dZ(0,0)^*(\xi,\eta)$ iff $\widehat{B}_0 = \widehat{A}_0 = 0$ and for $1 \leq k \leq N-1$

$$(\widehat{B}_k, \widehat{A}_k) = \left(\frac{1}{\sqrt{2\omega\left(\frac{k}{N}\right)}}(\xi_k + \eta_{N-k}),\ \frac{2}{\sqrt{2\omega\left(\frac{k}{N}\right)}}(e^{\mathrm{i}\pi k/N}\xi_k + e^{\mathrm{i}\pi(N-k)/N}\eta_{N-k}) \right) .$$

Thus there exist constants $C, C', C_Z > 0$, independent of N, such that

$$\|\underline{dZ}(0,0)^*(\xi,\eta)\|^2_{\mathcal{C}^{s+2,\sigma}} \leq \frac{C'}{N} \sum_{k=1}^{N-1} [k]_N^{2s}\, e^{2\sigma[k]_N} \omega\left(\tfrac{k}{N}\right) \frac{[k]_N^4}{\omega\left(\frac{k}{N}\right)^2} \left(|\xi_k|^2 + |\eta_k|^2\right) \leq C_Z^2 N^4 \, \|(\xi,\eta)\|^2_{\mathcal{P}^{s,\sigma}}$$

where in the last inequality we used that $[k]_N^4/\omega\left(\frac{k}{N}\right)^2 \leq C'' N^4$ for some constant $C'' > 0$ independent of N. Thus the second of (1.101) is proved.

Proof of property $(Z4)$. We will prove that Z is normally analytic. Recall that, as mentioned in the discussion before Proposition 1.47, the map Z is said to be normally analytic if $\check{Z} := Z\mathcal{F}$ is normally analytic. With an abuse of notations, we omit the “check” from Z.

We begin by expanding the components of Z, denoted by $Z_j(b,a) := (z_j(b,a),\, w_j(b,a))$, in Taylor series with center at $(b,a) = (0,0)$. The first two terms of the expansions are given by

$$\begin{aligned} z_j(b,a) &= D_j \langle L_p f_{2j,0}, \overline{f_{2j,0}} \rangle + D_j \langle L_p \left(L_0 - \lambda^0_{2j}\right)^{-1} (\mathbb{1} - P_{j0})\, L_p f_{2j,0}, \overline{f_{2j,0}} \rangle + O((b,a)^3), \\ w_j(b,a) &= D_j \langle L_p f_{2j-1,0}, \overline{f_{2j-1,0}} \rangle + D_j \langle L_p \left(L_0 - \lambda^0_{2j}\right)^{-1} (\mathbb{1} - P_{j0})\, L_p f_{2j-1,0}, \overline{f_{2j-1,0}} \rangle + O((b,a)^3). \end{aligned} \tag{1.159}$$

To perform the Taylor expansion at every order it is convenient to proceed in the following way. Write $z_j(b,a) = z_{j,1}(b,a) + z_{j,2}(b,a)$ and $w_j(b,a) = w_{j,1}(b,a) + w_{j,2}(b,a)$ where

$$z_{j,1}(b,a) = D_j \left\langle \left(L_0 - \lambda^0_{2j}\right) f_{2j}(b,a), \overline{f_{2j}(b,a)} \right\rangle, \quad z_{j,2}(b,a) = D_j \left\langle L_p f_{2j}(b,a), \overline{f_{2j}(b,a)} \right\rangle \ , \tag{1.160}$$

while $w_{j,1}(b,a)$ and $w_{j,2}(b,a)$ are defined as in (1.160), but with $f_{2j-1}(b,a)$ replacing $f_{2j}(b,a)$.

Expand $z_{j,\varsigma}(b,a)$, $\varsigma = 1,2$, in Taylor series with center at $(b,a) = (0,0)$: $z_{j,\varsigma}(b,a) = \sum_{n\geq 1} z^n_{j,\varsigma}(b,a)$, with $z^n_{j,\varsigma}$ a homogeneous polynomial of degree n in b, a. We write an analogous expansion for $w_{j,\varsigma}(b,a)$. Therefore one has

$$Z^n_j(b,a) := (z^n_j(b,a),\, w^n_j(b,a)) \equiv \left(z^n_{j,1}(b,a) + z^n_{j,2}(b,a),\, w^n_{j,1}(b,a) + w^n_{j,2}(b,a)\right).$$

In order to write explicitly $z^n_{j,\varsigma}(b,a)$ as a function of b and a, one needs to expand the vectors $f_{2j}(b,a)$ and $f_{2j-1}(b,a)$ in Taylor series of b, a. Rewrite (1.92), (1.97) as

$$f_{2j}(b,a) = U_j(b,a) f_{2j,0} = \left(\mathbb{1} - (P_j(b,a) - P_{j0})^2\right)^{-1/2} \Big(\mathbb{1} + (P_j(b,a) - P_{j0})\Big) f_{2j,0}$$

and expand the r.h.s. above in power series of $P_j(b,a) - P_{j0}$, getting:

$$f_{2j}(b,a) = \sum_{m=0}^{\infty} c_m \left(P_j(b,a) - P_{j0}\right)^m f_{2j,0}, \qquad f_{2j-1}(b,a) = \sum_{m=0}^{\infty} c_m \left(P_j(b,a) - P_{j0}\right)^m f_{2j-1,0} \ , \tag{1.161}$$

where the c_m's are the coefficients of the Taylor series of the function $\phi(x) = \frac{1+x}{(1-x^2)^{1/2}}$. Note that $c_{2k+1} = c_{2k} \equiv (-1)^k \binom{-1/2}{k}$, where $\binom{-1/2}{k} := -\frac{1}{2}(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-k+1)$ is the product of k negative terms, thus $(-1)^k\binom{-1/2}{k} \geq 0,\ \forall k \geq 0$, and therefore $c_m \geq 0,\ \forall m$.

By Corollary 1.51 (see also formula (1.156)) one has, in the ball $B^{\mathcal{C}^{s,\sigma}}(\epsilon_*/N^2)$,

$$P_j(b,a) - P_{j0} = \frac{\mathrm{i}}{2\pi}\sum_{n=1}^{\infty}(-1)^n \oint_{\Gamma_j} T^n(b,a,\lambda)\,(L_0-\lambda)^{-1}\,\mathrm{d}\lambda \tag{1.162}$$

where the Γ_j's are defined as in equation (1.91), and

$$T(b,a,\lambda) := (L_0-\lambda)^{-1}\,L_p\ .$$

Substituting (1.162) in (1.161) we get that

$$f_{2j}(b,a) = f_{2j,0} + \sum_{n\geq 1}\sum_{1\leq m\leq n} c_m \sum_{\alpha=(\alpha_1,\ldots,\alpha_m)\in\mathbb{N}^m,\,|\alpha|=n} f^{\alpha}_{2j,m}(b,a),$$

$$f^{\alpha}_{2j,m}(b,a) := \left(\frac{\mathrm{i}}{2\pi}\right)^m (-1)^{|\alpha|}\oint_{\Gamma_j}\ldots\oint_{\Gamma_j} T^{\alpha_1}(b,a,\lambda_1)\,(L_0-\lambda_1)^{-1}\ldots T^{\alpha_m}(b,a,\lambda_m)\,(L_0-\lambda_m)^{-1} f_{2j,0}\,\mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_m. \tag{1.163}$$

An analogous expansion holds for $f_{2j-1}(b,a)$, with $f_{2j-1,0}$ substituting $f_{2j,0}$ in the integral formula above. In order to write explicitly the expression inside the integral, one needs to compute the iterated terms $T^n(b,a,\lambda)f_{2j,0}$ and $T^n(b,a,\lambda)f_{2j-1,0}$. The computation turns out to be simpler if we express $L_p f_{2j,0}$ in the basis of the eigenvectors of L_0. To simplify the notations we relabel the eigenvectors of L_0 in the following way:

$$g_0 := f_{00},\quad g_N := f_{2N-1,0},\qquad g_j := f_{2j,0},\quad g_{-j} := f_{2j-1,0},\qquad \text{for } 1\leq j\leq N-1$$

and the eigenvalues of L_0 as

$$\hat\lambda_0 := \lambda^0_0,\quad \hat\lambda_N := \lambda^0_{2N-1},\qquad \hat\lambda_j := \lambda^0_{2j},\quad \hat\lambda_{-j} := \lambda^0_{2j-1},\qquad \text{for } 1\leq j\leq N-1.$$

For every $1\leq j\leq N-1$ one has that $\overline{g_j} = g_{-j}$, formally, one can also write $g_{j+2N} = g_j$, $\hat\lambda_j = \hat\lambda_{-j}$ and $\hat\lambda_{j+2N} = \hat\lambda_j$, as one verifies using the explicit expressions of the g_j's and $\hat\lambda_j$'s. In this notation, for $\lambda \neq \hat\lambda_{\pm j}$, one has $(L_0-\lambda)^{-1}g_{\pm j} = g_{\pm j}/(\hat\lambda_{\pm j}-\lambda)$. With a computation analogous to the one in (1.158) (using also the second formula in (1.143)), one verifies that the projection of $L_p g_j$ on the vector g_k is given by

$$\langle L_p\,g_j, g_k\rangle = \frac{1}{\sqrt{N}}\left(\hat b_{\frac{j-k}{2}} - 2\cos\left(\tfrac{k\pi}{N}\right)\hat a_{\frac{j-k}{2}}\right)\delta_{(j-k;\ \text{even}\ )}, \tag{1.164}$$

where $\delta_{(j-k;\ \text{even}\ )} = 1$ if $j-k$ is an even integer, and equals 0 otherwise. Formula (1.164) implies that $L_p g_j$ is supported only on the vectors g_k whose index k satisfies $k = j-2l$ for some integer l. Therefore we can write

$$T(b,a,\lambda)g_j = \sum_{l\in K^0_N}\frac{x^l_j}{\hat\lambda_{j-2l}-\lambda}g_{j-2l},\quad x^l_j := \langle L_p g_j, g_{j-2l}\rangle = \frac{1}{\sqrt{N}}\left(\hat b_l - 2\cos\left(\tfrac{(j-2l)\pi}{N}\right)\hat a_l\right)\ , \tag{1.165}$$

where K_N^0 is the set of indexes defined in (1.146). Note that $\left|x_j^l\right| \le \frac{2}{\sqrt{N}}\left(|\hat{b}_l| + |\hat{a}_l|\right)$ uniformly in j, and $x_j^{l+N} = x_j^l$. Iterating (1.165) one gets

$$T^n(b,a,\lambda)\,(L_0-\lambda)^{-1}\,g_j = \sum_{i_1,\cdots,i_n\in K_N^0} \frac{x_j^{i_1}x_{j-2i_1}^{i_2}\dots x_{j-2i_1-\cdots-2i_{n-1}}^{i_n}}{(\hat{\lambda}_j-\lambda)\prod_{l=1}^n\left(\hat{\lambda}_{j-2\sum_{m=1}^l i_m}-\lambda\right)} g_{j-2i_1-\cdots-2i_n}.$$

More generally, for a vector $\alpha = (\alpha_1,\dots,\alpha_m)\in\mathbb{N}^m$ with $|\alpha| = n$ and $\lambda_1,\cdots,\lambda_m\in\Gamma_j$, one has

$$T^{\alpha_m}(b,a,\lambda_m)\,(L_0-\lambda_m)^{-1}\cdots T^{\alpha_1}(b,a,\lambda_1)\,(L_0-\lambda_1)^{-1}\,g_j =$$

$$= \sum_{i_1,\dots,i_n\in K_N^0} \frac{x_j^{i_1}x_{j-2i_1}^{i_2}\dots x_{j-2i_1-\cdots-2i_{n-1}}^{i_n}}{(\hat{\lambda}_j-\lambda_1)\prod_{l=1}^n\left(\hat{\lambda}_{j-2\sum_{m=1}^l i_m}-\mu_l\right)\prod_{l=1}^{m-1}\left(\hat{\lambda}_{j-2\sum_{h=1}^{\alpha_1+\cdots+\alpha_l} i_h}-\lambda_{l+1}\right)} g_{j-2i_1-\cdots-2i_n} \tag{1.166}$$

where

$$\mu_l = \lambda_1 \text{ for } 1\le l\le\alpha_1, \quad \text{and} \quad \mu_l = \lambda_k \text{ for } \sum_{h=1}^{k-1}\alpha_h + 1 \le l \le \sum_{h=1}^{k}\alpha_h, \quad 2\le k\le m\,. \tag{1.167}$$

To obtain the explicit expression of $z_{j,\varsigma}^n$ and $w_{j,\varsigma}^n$, $\varsigma = 1,2$, in terms of the Fourier variables $\hat{b}$, $\hat{a}$, we substitute (1.166) in (1.163) and the obtained result in (1.160). By (1.163), $z_{j,1}^n$ is a sum of terms of the form $\left\langle\left(L_0-\lambda_{2j}^0\right)f_{2j,p_1}^{\alpha}, \overline{f_{2j,p_2}^{\beta}}\right\rangle$ over $(p,\alpha,\beta)\in\mathbb{N}^2\times\mathbb{N}^{p_1}\times\mathbb{N}^{p_2}$ with $|p| = p_1+p_2\le n$ and $|\alpha|+|\beta| = n$. For $|\alpha| = r$, $|\beta| = n-r$ one gets

$$\begin{aligned}\left\langle\left(L_0-\hat{\lambda}_j\right)f_{2j,p_1}^{\alpha}, \overline{f_{2j,p_2}^{\beta}}\right\rangle &= \left(\frac{\mathrm{i}}{2\pi}\right)^{|p|}(-1)^n\oint_{\Gamma_j}\dots\oint_{\Gamma_j}\kappa_{j,1}^{p,\alpha,\beta}(i)x_j^{i_1}x_{j-2i_1}^{i_2}\dots x_{j-2i_1-\cdots-2i_{r-1}}^{i_r}\times\\ &\times x_j^{i_n}x_{j-2i_n}^{i_{n-1}}\dots x_{j-2i_n-\cdots-2i_{r+2}}^{i_{r+1}}\left\langle g_{j-2i_1-\cdots-2i_r}, \overline{g_{j-2i_{r+1}-\cdots-2i_n}}\right\rangle \mathrm{d}\lambda_1\dots\mathrm{d}\lambda_{|p|},\end{aligned} \tag{1.168}$$

where, writing $\mathbf{i} = (i_1,\cdots,i_n)$,

$$\begin{aligned}\kappa_{j,1}^{p,\alpha,\beta}(\mathbf{i}) &:= \frac{\left(\hat{\lambda}_{j-2\sum_{m=1}^r i_m}-\hat{\lambda}_j\right)}{(\hat{\lambda}_j-\lambda_1)\prod_{l=1}^r\left(\hat{\lambda}_{j-2\sum_{m=1}^l i_m}-\mu_l\right)\prod_{l=1}^{p_1-1}\left(\hat{\lambda}_{j-2\sum_{h=1}^{\alpha_1+\cdots+\alpha_l} i_h}-\lambda_{l+1}\right)}\times\\ &\times\frac{1}{(\hat{\lambda}_j-\lambda_{p_1+1})\prod_{l=r+1}^n\left(\hat{\lambda}_{j-2\sum_{m=l}^n i_m}-\tilde{\mu}_l\right)\prod_{l=1}^{p_2-1}\left(\hat{\lambda}_{j-2\sum_{h=1}^{\beta_1+\cdots+\beta_l} i_h}-\lambda_{l+1}\right)}\end{aligned} \tag{1.169}$$

and the $\tilde{\mu}_l$'s are defined as in (1.167), but with the multi-index β replacing α. Similarly, the term $z_{j,2}^n$ is a sum of terms of the form $\left\langle L_p\,f_{2j,p_1}^{\alpha}, \overline{f_{2j,p_2}^{\beta}}\right\rangle$ over $(p,\alpha,\beta)\in\mathbb{N}^2\times\mathbb{N}^{p_1}\times\mathbb{N}^{p_2}$ with $|p|\le n$ and $|\alpha|+|\beta| = n-1$. The term $\left\langle L_p\,f_{2j,p_1}^{\alpha}, \overline{f_{2j,p_2}^{\beta}}\right\rangle$ has an expression similar to (1.168), and for $|\alpha| = r$ and $|\beta| = n-1-r$ the kernel $\kappa_{j,2}^{p,\alpha,\beta}(\mathbf{i})$ is given by

$$\begin{aligned}\kappa_{j,2}^{p,\alpha,\beta}(\mathbf{i}) &:= \frac{1}{(\hat{\lambda}_j-\lambda_1)\prod_{l=1}^r\left(\hat{\lambda}_{j-2\sum_{m=1}^l i_m}-\mu_l\right)\prod_{l=1}^{p_1-1}\left(\hat{\lambda}_{j-2\sum_{h=1}^{\alpha_1+\cdots+\alpha_l} i_h}-\lambda_{l+1}\right)}\times\\ &\times\frac{1}{(\hat{\lambda}_j-\lambda_{p_1+1})\prod_{l=r+2}^n\left(\hat{\lambda}_{j-2\sum_{m=l}^n i_m}-\tilde{\mu}_l\right)\prod_{l=1}^{p_2-1}\left(\hat{\lambda}_{j-2\sum_{h=1}^{\beta_1+\cdots+\beta_l} i_h}-\lambda_{l+1}\right)}\,.\end{aligned} \tag{1.170}$$

Using the explicit form of the eigenvectors $\{g_k\}_{-(N-1)\leq k\leq N}$ (see Lemma 1.48), one verifies that

$$\left\langle g_{j-2i_1-\cdots-2i_r}, \overline{g_{j-2i_{r+1}-\cdots-2i_n}}\right\rangle = \delta\left(j, \sum_{m=1}^n i_m\right), \quad \left\langle g_{N-j-2i_1-\cdots-2i_r}, \overline{g_{N-j-2i_{r+1}-\cdots-2i_n}}\right\rangle = \delta\left(-j, \sum_{m=1}^n i_m\right).$$

This is used to simplify the last term in (1.168). Moreover, using $j = \sum_{m=1}^n i_m$ and the identity $\hat{\lambda}_j = \hat{\lambda}_{-j}$, one gets that

$$\hat{\lambda}_{j-2i_n} = \hat{\lambda}_{j-2\sum_{m=1}^{n-1} i_m}, \quad \ldots, \quad \hat{\lambda}_{j-2i_n-2i_{n-1}-\cdots-2i_{r+1}} = \hat{\lambda}_{j-2\sum_{m=1}^{r} i_m}. \tag{1.171}$$

Recalling the definition of the coefficients x_j^l (formula (1.165)), we can write, for $\varsigma = 1, 2$,

$$z_{j,\varsigma}^n(\hat{b}, \hat{a}) = \frac{1}{N^{n/2}} \left(\tfrac{2}{N}\omega\left(\tfrac{j}{N}\right)\right)^{-1/2} \sum_{(\mathbf{i},\boldsymbol{\iota})\in\Delta^n} \mathcal{K}_{j,\varsigma}^n(\mathbf{i},\boldsymbol{\iota})\, u_{i_1,\iota_1}\ldots u_{i_n,\iota_n} \tag{1.172}$$

where the set

$$\Delta^n := \left\{(\mathbf{i},\boldsymbol{\iota}) \in \mathbb{Z}^n \times \mathbb{N}^n : \, i_l \in K_N^0, \quad \iota_l \in \{1,2\}, \quad \forall 1 \leq l \leq n\right\},$$

the variables $u = (u_{i_1,\iota_1}, \cdots, u_{i_n,\iota_n})$ are defined by

$$u_{i_r,1} := \hat{b}_{i_r}, \qquad u_{i_r,2} := \hat{a}_{i_r},$$

the kernels $\mathcal{K}_{j,\varsigma}^n(\mathbf{i},\boldsymbol{\iota})$ are defined for $(\mathbf{i},\boldsymbol{\iota}) \in \Delta^n$ by

$$\mathcal{K}_{j,\varsigma}^n(\mathbf{i},\boldsymbol{\iota}) := \tilde{\mathcal{K}}_{j,\varsigma}^n(\mathbf{i}) \prod_{\{1\leq l\leq n\}} \left(-2\cos\left(\tfrac{(j-2i_i-\cdots-2i_l)\pi}{N}\right)\right)^{\iota_l-1}, \tag{1.173}$$

$$\tilde{\mathcal{K}}_{j,\varsigma}^n(\mathbf{i}) = \sum_{\substack{r+s=n-(\varsigma-1)\\ p=(p_1,p_2)\in\mathbb{N}^2,\, |p|\leq n}} c_{p_1}c_{p_2} \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^{p_1}\times\mathbb{N}^{p_2}\\ |\alpha|=r,\,|\beta|=s}} \mathcal{S}_{j,\varsigma}^{p,\alpha,\beta}(\mathbf{i}) \tag{1.174}$$

and finally

$$\mathcal{S}_{j,\varsigma}^{p,\alpha,\beta}(\mathbf{i}) = \delta\left(j, \sum_{m=1}^n i_m\right)\left(\frac{\mathrm{i}}{2\pi}\right)^{|p|}(-1)^n \oint_{\Gamma_j}\cdots\oint_{\Gamma_j} \kappa_{j,\varsigma}^{p,\alpha,\beta}(\mathbf{i})\,\mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_{|p|}. \tag{1.175}$$

An analogous expansion holds also for $w_{j,1}^n$ and $w_{j,2}^n$.
We need now to get estimates of the kernels $\mathcal{K}_{j,\varsigma}^n$, which will follow from estimates on the denominators of $\kappa_{j,\varsigma}^{p,\alpha,\beta}$.

Lemma 1.73. *Let* $\mu \in \Gamma_j := \left\{\lambda \in \mathbb{C} : \, \left|\lambda - \lambda_{2j}^0\right| = \min\left(\frac{\langle j\rangle}{2N^2}, \frac{\langle N-j\rangle}{2N^2}\right)\right\}$, *where* $\langle j\rangle = \left(1+|j|^2\right)^{1/2}$. *Then there exists a constant* $R > 0$, *independent of* N, *such that for every* $-(N-1) \leq k \leq N$ *one has*

$$\left|\hat{\lambda}_k - \mu\right| \geq \begin{cases} R\langle j-k\rangle\langle j+k\rangle/N^2, & \text{if } 0 \leq |j| \leq \lfloor N/2\rfloor \\ R\langle j-k\rangle\langle (N-j)+(N-k)\rangle/N^2, & \text{if } \lfloor N/2\rfloor + 1 \leq |j| \leq N \end{cases} \tag{1.176}$$

Proof. Consider first the situation in which both the eigenvalues $\hat{\lambda}_j$ and $\hat{\lambda}_k$ are in the low half of the spectrum, namely $0 \le |j|, |k| \le \lfloor N/2 \rfloor$. In this case one has

$$|\hat{\lambda}_k - \hat{\lambda}_j| \equiv |\lambda^0_{2|k|} - \lambda^0_{2|j|}| = 2\left|\cos\left(\tfrac{|k|\pi}{N}\right) - \cos\left(\tfrac{|j|\pi}{N}\right)\right| = 2\left|\cos\left(\tfrac{k\pi}{N}\right) - \cos\left(\tfrac{j\pi}{N}\right)\right| \ge \frac{4|j^2 - k^2|}{N^2}.$$

Therefore, for $k \ne j$, there exists a positive constant R_1 such that for $\forall \mu \in \Gamma_j$

$$\left|\hat{\lambda}_k - \mu\right| \ge \left|\hat{\lambda}_k - \hat{\lambda}_j\right| - \frac{\langle j \rangle}{2N^2} \ge \frac{4|j^2 - k^2|}{N^2} - \frac{\langle j \rangle}{2N^2} \ge R_1 \frac{\langle j - k \rangle \langle j + k \rangle}{N^2}, \tag{1.177}$$

where we used the inequality $\langle j \rangle \le 2 \langle j - k \rangle \langle j + k \rangle$, which holds since j, k are integers. If $k = j$, then the claimed estimate follows trivially since $|\hat{\lambda}_k - \mu| = \langle j \rangle / 2N^2$.
Consider now the case when $\hat{\lambda}_j$ is in the low half of the spectrum, while $\hat{\lambda}_k$ is in the high half, i.e. $0 \le |j| \le \lfloor N/2 \rfloor$, while $\lfloor N/2 \rfloor < |k| \le N$. In this case the distance of the eigenvalues $\hat{\lambda}_j$ and $\hat{\lambda}_k$ is of order $\frac{1}{N}$, therefore the estimate (1.176) holds as well. More precisely, using $\cos x \ge 1 - \frac{2}{\pi} x$ for $0 \le x \le \pi/2$, one has

$$|\hat{\lambda}_k - \hat{\lambda}_j| = |\lambda^0_{2|k|} - \lambda^0_{2|j|}| = 2\left|\cos\left(\tfrac{(N-|k|)\pi}{N}\right) + \cos\left(\tfrac{j\pi}{N}\right)\right| \ge \frac{4(|k| - |j|)}{N} \ge \frac{\langle j - k \rangle \langle j + k \rangle}{N^2},$$

where the last inequality holds since $\langle l \rangle / N \le 4, \ \forall |l| \le 2N$. The inequality above implies that

$$\left|\hat{\lambda}_k - \mu\right| \ge |\hat{\lambda}_k - \hat{\lambda}_j| - \frac{\langle j \rangle}{2N^2} \ge \frac{\langle j - k \rangle \langle j + k \rangle}{N^2} - \frac{\langle j \rangle}{2N^2} \ge R_2 \frac{\langle j - k \rangle \langle j + k \rangle}{N^2}, \tag{1.178}$$

for some $R_2 > 0$. Thus the first of (1.176) is proved.
The proof of the second inequality of (1.176) follows by symmetry and is omitted. □

We can now estimate the kernels $\mathcal{K}^n_{j,\varsigma}$ defined in (1.173).

Lemma 1.74. *There exists a constant $R > 0$, independent of N, such that $\mathcal{K}^n_{j,\varsigma}(\mathbf{i}, \boldsymbol{\iota})$, $\varsigma = 1, 2$, satisfy, for every $n \ge 2$ and $1 \le j \le \lfloor N/2 \rfloor$, the estimates*

$$\begin{aligned}
\left|\mathcal{K}^n_{j,\varsigma}(\mathbf{i}, \boldsymbol{\iota})\right| &\le R^n N^{2(n-1)} \delta\left(j, \sum_{l=1}^n i_l\right) \frac{1}{\prod_{l=1}^{n-1} \left\langle \sum_{k=1}^l i_k \right\rangle \left\langle \sum_{k=1}^l i_k - j \right\rangle}, \\
\left|\mathcal{K}^n_{N-j,\varsigma}(\mathbf{i}, \boldsymbol{\iota})\right| &\le R^n N^{2(n-1)} \delta\left(-j, \sum_{l=1}^n i_l\right) \frac{1}{\prod_{l=1}^{n-1} \left\langle \sum_{k=1}^l i_k \right\rangle \left\langle \sum_{k=1}^l i_k - j \right\rangle}.
\end{aligned} \tag{1.179}$$

Proof. We start by estimating $\kappa^{p,\alpha,\beta}_{j,\varsigma}(\mathbf{i})$, defined in (1.169) and (1.170). For every $-(N-1) \le k \le N$ and $\mu \in \Gamma_j$ one has $\left|\hat{\lambda}_k - \mu\right| \ge \left|\hat{\lambda}_j - \mu\right| \ge \min\left(\frac{\langle j \rangle}{2N^2}, \frac{\langle N-j \rangle}{2N^2}\right)$, therefore

$$\begin{aligned}
&\left|(\hat{\lambda}_j - \lambda_1) \prod_{l=1}^{p_1 - 1} \left(\hat{\lambda}_{j - 2\sum_{h=1}^{\alpha_1 + \dots + \alpha_l} i_h} - \lambda_{l+1}\right) (\hat{\lambda}_j - \lambda_{p_1 + 1}) \prod_{l=1}^{p_2 - 1} \left(\hat{\lambda}_{j - 2\sum_{h=1}^{\beta_1 + \dots + \beta_l} i_h} - \lambda_{l+1}\right)\right| \\
&\qquad \ge \left[\min\left(\frac{\langle j \rangle}{2N^2}, \frac{\langle N - j \rangle}{2N^2}\right)\right]^{|p|}.
\end{aligned}$$

Let now $1 \le j \le \lfloor N/2 \rfloor$. By Lemma 1.73, formula (1.171) and the inequality $\frac{|\hat{\lambda}_{j-2\sum_{m=1}^r i_m} - \hat{\lambda}_j|}{|\hat{\lambda}_{j-2\sum_{m=1}^r i_m} - \mu|} \le 2$ (which is used to estimate just $\kappa_{j,1}^{p,\alpha,\beta}(\mathbf{i})$), it follows that, for $\varsigma = 1, 2$,

$$\left|\kappa_{j,\varsigma}^{p,\alpha,\beta}(\mathbf{i})\right| \le \frac{2}{\left[\min\left(\frac{\langle j\rangle}{2N^2}, \frac{\langle N-j\rangle}{2N^2}\right)\right]^{|p|} \prod_{l=1}^{n-1}\left|\hat{\lambda}_{j-2\sum_{m=1}^l i_m} - \mu_l\right|} \le \frac{2\, a_j(i_1, \cdots, i_{n-1})}{\left[\min\left(\frac{\langle j\rangle}{2N^2}, \frac{\langle N-j\rangle}{2N^2}\right)\right]^{|p|}}$$

where

$$a_j(i_1, \cdots, i_{n-1}) := \frac{R^{n-1}N^{2(n-1)}}{\prod_{l=1}^{n-1}\left\langle \sum_{k=1}^l i_k \right\rangle \left\langle \sum_{k=1}^l i_k - j \right\rangle} .$$

To estimate $\mathcal{S}_{j,\varsigma}^{p,\alpha,\beta}$ consider (1.175). The $\mathcal{S}_{j,\varsigma}^{p,\alpha,\beta}$'s are defined by integrating the kernels $\kappa_{j,\varsigma}^{p,\alpha,\beta}$ over Γ_j $|p|$-times. Since $|\Gamma_j| = 2\pi \min\left(\frac{\langle j\rangle}{2N^2}, \frac{\langle N-j\rangle}{2N^2}\right)$, one gets

$$\left|\mathcal{S}_{j,\varsigma}^{p,\alpha,\beta}(\mathbf{i})\right| \le \left[\min\left(\frac{\langle j\rangle}{N^2}, \frac{\langle N-j\rangle}{N^2}\right)\right]^{|p|} \delta\left(j, \sum_{l=1}^n i_l\right)\left|\kappa_{j,\varsigma}^{p,\alpha,\beta}(\mathbf{i})\right| \le 2\delta\left(j, \sum_{l=1}^n i_l\right) a_j(i_1, \cdots, i_{n-1}).$$

Finally consider $\mathcal{K}_{j,\varsigma}^n$. From (1.173) one has $\left|\mathcal{K}_{j,\varsigma}^n(\mathbf{i}, \boldsymbol{\iota})\right| \le 2^n \left|\tilde{\mathcal{K}}_{j,\varsigma}^n(\mathbf{i})\right|$, and from (1.174)

$$\begin{aligned}\left|\tilde{\mathcal{K}}_{j,\varsigma}^n(\mathbf{i})\right| &\le \delta\left(j, \sum_{l=1}^n i_l\right) a_j(i_1, \cdots, i_{n-1}) \sum_{\substack{r+s=n-(\varsigma-1)\\ p=(p_1,p_2)\in\mathbb{N}^2,\, |p|\le n}} c_{p_1}c_{p_2} \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^{p_1}\times\mathbb{N}^{p_2}\\ |\alpha|=r,\, |\beta|=s}} 1\\ &\le C^n\delta\left(j, \sum_{l=1}^n i_l\right) a_j(i_1, \cdots, i_{n-1}) ,\end{aligned}$$

thus the first estimate of (1.179) follows. The proof of the second one is similar, and is omitted. □

Define now $\mathcal{K}_j^n := \mathcal{K}_{j,1}^n + \mathcal{K}_{j,2}^n$. Then

$$z_j^n(\hat{b}, \hat{a}) = \frac{D_j}{N^{n/2}} \sum_{(\mathbf{i},\boldsymbol{\iota})\in\Delta^n} \mathcal{K}_j^n(\mathbf{i}, \boldsymbol{\iota})\, u_{i_1,\iota_1} \dots u_{i_n,\iota_n}, \quad w_j^n(\hat{b}, \hat{a}) = \frac{D_j}{N^{n/2}} \sum_{(\mathbf{i},\boldsymbol{\iota})\in\Delta^n} \mathcal{H}_j^n(\mathbf{i}, \boldsymbol{\iota})\, u_{i_1,\iota_1} \dots u_{i_n,\iota_n}, \tag{1.180}$$

where $\mathcal{H}_j^n(\mathbf{i}, \boldsymbol{\iota}) = \overline{\mathcal{K}_j^n(-\mathbf{i}, \boldsymbol{\iota})}$. The second formula holds since for b, a real one has $w^n(b, a) = \overline{z^n(b, a)}$.

Corollary 1.75. *Let $\Delta_j^n := \{(\mathbf{i}, \boldsymbol{\iota}) \in \Delta^n : \sum_{l=1}^n i_l = j\}$. Then for $1 \le j \le \lfloor N/2 \rfloor$ one has supp $\mathcal{K}_j^n \subseteq \Delta_j^n$ and supp $\mathcal{K}_{N-j}^n \subseteq \Delta_{-j}^n$. Moreover*

$$\left\|\mathcal{K}_j^n\right\|_{\Delta_j^n}, \quad \left\|\mathcal{K}_{N-j}^n\right\|_{\Delta_{-j}^n} \le \frac{R^n N^{2(n-1)}}{\langle j\rangle^{n-1}}, \tag{1.181}$$

where $\left\|\mathcal{K}_j^n\right\|_{\Delta_j^n}^2 := \sup_{\iota_1,\cdots,\iota_n\in\{1,2\}} \sum_{i_1+\cdots+i_n=j} \left|\mathcal{K}_j^n(\mathbf{i}, \boldsymbol{\iota})\right|^2$.

Proof. Just remark that $\frac{\langle j\rangle^2}{\langle k\rangle^2\langle k-j\rangle^2} \le 4\left(\frac{1}{\langle k\rangle^2} + \frac{1}{\langle k-j\rangle^2}\right)$. $\square$

We prove now bounds on the map $\underline{Z^n}(\hat{b},\hat{a}) := (\underline{z^n}(\hat{b},\hat{a}), \underline{w^n}(\hat{b},\hat{a}))$.

Lemma 1.76. *There exists a constant $C>0$, independent of N, such that for any $s\ge 0$ and $\sigma\ge 0$*

$$\left\|\underline{Z}^n(|\hat{b}|,|\hat{a}|)\right\|_{\mathcal{P}^{s+1,\sigma}} \le C^n N^{2(n-1)} \|(b,a)\|^n_{\mathcal{C}^{s,\sigma}}, \qquad \forall\, n\ge 2. \tag{1.182}$$

Proof. By formula (1.172) one has that for $1\le j\le \lfloor N/2\rfloor$

$$\begin{aligned}
\left|\underline{z}^n_j(|\hat{b}|,|\hat{a}|)\right| &\le \frac{D_j}{N^{n/2}} \sum_{(\mathbf{i},\boldsymbol{\iota})\in\Delta^n_j} \left|\mathcal{K}^n_j(\mathbf{i},\boldsymbol{\iota})\right| |u_{i_1,\iota_1}|\ldots|u_{i_n,\iota_n}|,\\
\left|\underline{z}^n_{N-j}(|\hat{b}|,|\hat{a}|)\right| &\le \frac{D_j}{N^{n/2}} \sum_{(\mathbf{i},\boldsymbol{\iota})\in\Delta^n_{-j}} \left|\mathcal{K}^n_{N-j}(\mathbf{i},\boldsymbol{\iota})\right| |u_{i_1,\iota_1}|\ldots|u_{i_n,\iota_n}|.
\end{aligned}\tag{1.183}$$

Introduce $\Lambda(\mathbf{i}) := [i_1]\cdots[i_n]$, where $[i_r] = \max(1,|i_r|)$ $\forall 1\le r\le n$, and remark that for some constant $R>0$ one has

$$\sup_{i_1+\cdots+i_n=j} \Lambda(\mathbf{i})^{-1} \le \frac{R^n}{\langle j\rangle}, \qquad \forall j\in\mathbb{Z}.$$

Therefore, by Corollary 1.75,

$$\left|\underline{z}^n_j(|\hat{b}|,|\hat{a}|)\right|^2 \le \frac{1}{N^n} D_j^2 \left\|\mathcal{K}^n_j\right\|^2_{\Delta^n_j} \left(\sup_{i_1+\cdots+i_n=j}\Lambda(\mathbf{i})^{-2s}\right) \sum_{(\mathbf{i},\boldsymbol{\iota})\in\Delta^n_j} [i_1]^{2s}|u_{i_1,\iota_1}|^2\ldots[i_n]^{2s}|u_{i_n,\iota_n}|^2,$$

$$\left|\underline{z}^n_{N-j}(|\hat{b}|,|\hat{a}|)\right|^2 \le \frac{1}{N^n} D_j^2 \left\|\mathcal{K}^n_{N-j}\right\|^2_{\Delta^n_{-j}} \left(\sup_{i_1+\cdots+i_n=-j}\Lambda(\mathbf{i})^{-2s}\right) \sum_{(\mathbf{i},\boldsymbol{\iota})\in\Delta^n_{-j}} [i_1]^{2s}|u_{i_1,\iota_1}|^2\ldots[i_n]^{2s}|u_{i_n,\iota_n}|^2.$$

Use now inequalities (1.181), the definition of D_j, the fact that $e^{2\sigma|j|}\le e^{2\sigma|i_1|}\ldots e^{2\sigma|i_{n-1}|}e^{2\sigma|j-i_1-\cdots-i_{n-1}|}$, and the bounds $|u_{l,\iota_l}|\le|\hat{b}_l|+|\hat{a}_l|$, to deduce that, for any $n\ge 2$,

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{\lfloor N/2\rfloor}[j]^{2(s+1)}e^{2\sigma|j|}\omega\left(\tfrac{j}{N}\right)\left(\left|\underline{z}^n_j(|\hat{b}|,|\hat{a}|)\right|^2 + \left|\underline{z}^n_{N-j}(|\hat{b}|,|\hat{a}|)\right|^2\right)\\
&\le N^{4(n-1)}\frac{C^n}{N^n}\sum_{j=1}^{\lfloor N/2\rfloor}[j]^{2(2-n)}e^{2\sigma|j|}\sum_{(\mathbf{i},\boldsymbol{\iota})\in\Delta^n_{\pm j}}[i_1]^{2s}|u_{i_1,\iota_1}|^2\ldots[i_n]^{2s}|u_{i_n,\iota_n}|^2\\
&\le N^{4(n-1)}C^n\|(b,a)\|^{2n}_{\mathcal{C}^{s,\sigma}}.
\end{aligned}$$

Since $w^n(\hat{b},\hat{a})$ satisfies the same inequality, estimate (1.182) holds. $\square$

Consider now the map $(\hat{b},\hat{a})\mapsto dZ^n(\hat{b},\hat{a})^*$, where $dZ^n(\hat{b},\hat{a})^*$ is the adjoint of the differential of Z^n. Explicitly, if ξ,η are vectors in $\mathbb{C}^{N-1}$ and h,g are vectors in $\mathbb{C}^N$ such that $(h,g)\equiv dZ^n(\hat{b},\hat{a})^*(\xi,\eta)$, then the j^{th} components of h and g are given by

$$(h_j,\,g_j) = \left(\sum_{k=1}^{N-1}\left(\overline{\frac{\partial z^n_k}{\partial\hat{b}_j}(\hat{b},\hat{a})}\xi_k + \overline{\frac{\partial w^n_k}{\partial\hat{b}_j}(\hat{b},\hat{a})}\eta_k\right),\quad \sum_{k=1}^{N-1}\left(\overline{\frac{\partial z^n_k}{\partial\hat{a}_j}(\hat{b},\hat{a})}\xi_k + \overline{\frac{\partial w^n_k}{\partial\hat{a}_j}(\hat{b},\hat{a})}\eta_k\right)\right). \tag{1.184}$$

Denote by $\underline{h}, \underline{g}$ the vectors of $\mathbb{C}^N$ whose components are given by

$$(\underline{h_j},\, \underline{g_j}) = \left(\sum_{k=1}^{N-1} \left(\underline{\frac{\partial z_k^n}{\partial \hat{b}_j}}(|\hat{b}|, |\hat{a}|)|\xi_k| + \underline{\frac{\partial w_k^n}{\partial \hat{b}_j}}(|\hat{b}|, |\hat{a}|)|\eta_k| \right), \quad \sum_{k=1}^{N-1} \left(\underline{\frac{\partial z_k^n}{\partial \hat{a}_j}}(|\hat{b}|, |\hat{a}|)|\xi_k| + \underline{\frac{\partial w_k^n}{\partial \hat{a}_j}}(|\hat{b}|, |\hat{a}|)|\eta_k| \right) \right). \tag{1.185}$$

We begin to study the case $n = 2$.

Lemma 1.77. *There exists a constant $R > 0$, independent of N, such that $\forall s \geq 0$, $\sigma \geq 0$ one has*

$$\left\| \underline{dZ^2}(|\hat{b}|, |\hat{a}|)^*(|\xi|, |\eta|) \right\|_{\mathcal{C}^{s+2,\sigma}} \leq RN^3 \left\| (b, a) \right\|_{\mathcal{C}^{s,\sigma}} \left\| (\xi, \eta) \right\|_{\mathcal{P}^{s,\sigma}} . \tag{1.186}$$

Proof. By (1.159), one computes that the second order terms $Z^2 = (z^2, w^2)$ are given by

$$z_k^2(\hat{b}, \hat{a}) = \tfrac{D_k}{N} \sum_{l \neq 0} \left(\hat{b}_l - 2\cos(\tfrac{(k-2l)\pi}{N})\hat{a}_l \right) \left(\hat{b}_{k-l} - 2\cos(\tfrac{k\pi}{N})\hat{a}_{k-l} \right) / (\lambda^0_{2(k-2l)} - \lambda^0_{2k})$$
$$w_k^2(\hat{b}, \hat{a}) = \tfrac{D_k}{N} \sum_{l \neq 0} \left(\hat{b}_{N-l} - 2\cos(\tfrac{(k-2l)\pi}{N})\hat{a}_{N-l} \right) \left(\hat{b}_{l-k} - 2\cos(\tfrac{k\pi}{N})\hat{a}_{l-k} \right) / (\lambda^0_{2(k-2l)} - \lambda^0_{2k}).$$

Let $\underline{h}$, $\underline{g}$ be as in (1.185) with $n = 2$. Using the explicit expressions for z_k^2 and w_k^2, one computes that for $0 \leq j \leq \lfloor N/2 \rfloor$

$$\begin{aligned}
|\underline{h_j}| &\leq \frac{1}{N} \sum_{k=1}^{N-1} \frac{\left(|\hat{b}_{k-j}| + 2|\hat{a}_{k-j}|\right) D_k(|\xi_k| + |\eta_k|)}{|\lambda^0_{2(k-2j)} - \lambda^0_{2k}|} \\
&\leq N \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{\left(|\hat{b}_{k-j}| + 2|\hat{a}_{k-j}|\right) D_k(|\xi_k| + |\eta_k|)}{\langle k-j \rangle \langle j \rangle} + N \sum_{k=\lfloor N/2 \rfloor+1}^{N-1} \frac{\left(|\hat{b}_{k-j}| + 2|\hat{a}_{k-j}|\right) D_k(|\xi_k| + |\eta_k|)}{\langle N-k+j \rangle \langle j \rangle} \\
&\leq N \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{\left(|\hat{b}_{k-j}| + 2|\hat{a}_{k-j}|\right) D_k(|\xi_k| + |\eta_k|)}{\langle k-j \rangle \langle j \rangle} + \frac{\left(|\hat{b}_{N-k-j}| + 2|\hat{a}_{N-k-j}|\right) D_k(|\xi_{N-k}| + |\eta_{N-k}|)}{\langle k+j \rangle \langle j \rangle} \\
&\leq \frac{N^2}{\langle j \rangle} \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{\left(|\hat{b}_{k-j}| + 2|\hat{a}_{k-j}|\right) \langle k \rangle^{1/2}(|\xi_k| + |\eta_k|)}{\langle k-j \rangle \langle k \rangle} + \frac{\left(|\hat{b}_{N-k-j}| + 2|\hat{a}_{N-k-j}|\right) \langle k \rangle^{1/2}(|\xi_{N-k}| + |\eta_{N-k}|)}{\langle k+j \rangle \langle k \rangle}
\end{aligned}$$

where in the last inequality we used that $D_k \leq N/\langle k \rangle^{1/2}$. With analogous computations, one verifies that

$$|\underline{h_{N-j}}| \leq \frac{N^2}{\langle j \rangle} \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{\left(|\hat{b}_{k+j}| + 2|\hat{a}_{k+j}|\right) \langle k \rangle^{1/2}(|\xi_k| + |\eta_k|)}{\langle k+j \rangle \langle k \rangle} + \frac{\left(|\hat{b}_{j-k}| + 2|\hat{a}_{j-k}|\right) \langle k \rangle^{1/2}(|\xi_{N-k}| + |\eta_{N-k}|)}{\langle k-j \rangle \langle k \rangle}.$$

Proceeding as in the proof of Lemma 1.70, one obtains that there exist constants $C, C' > 0$, independent of N, such that

$$
\begin{aligned}
&\frac{1}{N}\sum_{j=0}^{\lfloor N/2\rfloor}[j]^{2(s+2)}e^{2\sigma|j|}(|\underline{h_j}|^2+|\underline{h_{N-j}}|^2)\\
&\quad\leq CN^3\left(\sum_{k=0}^{N-1}[k]_N^{2s}e^{2\sigma[k]_N}(|\hat{b}_k|^2+|\hat{a}_k|^2)\right)\left(\sum_{l=1}^{N-1}[l]_N^{2s}e^{2\sigma[l]_N}[l]_N(|\xi_l|^2+|\eta_l|^2)\right)\\
&\quad\leq C'N^6\left\|(b,a)\right\|^2_{\mathcal{C}^{s,\sigma}}\left\|(\xi,\eta)\right\|^2_{\mathcal{P}^{s,\sigma}} \qquad (1.187)
\end{aligned}
$$

where in the last inequality we used that $[l]_N \leq N\omega\left(\frac{l}{N}\right)$ for l integer. One verifies that $\underline{g}$ satisfies the same inequality as (1.187). Thus estimate (1.186) follows from the following inequality:

$$
\left\|\underline{dZ^2}(|\hat{b}|,|\hat{a}|)^*(|\xi|,|\eta|)\right\|^2_{\mathcal{C}^{s+2,\sigma}}\leq\frac{1}{N}\sum_{j=0}^{N-1}[j]_N^{2s+4}e^{2\sigma[j]_N}\left(|\underline{h_j}|^2+|\underline{g_j}|^2\right). \qquad (1.188)
$$

□

We study now $dZ^n(\hat{b},\hat{a})^*$ for $n\geq 3$.

Lemma 1.78. *There exists a constant $R>0$, independent of N, such that for every $s\geq 0$, $\sigma\geq 0$ and $n\geq 3$*

$$
\left\|\underline{dZ^n}(|\hat{b}|,|\hat{a}|)^*(|\xi|,|\eta|)\right\|_{\mathcal{C}^{s+2,\sigma}}\leq R^nN^{2n-1}\left\|(b,a)\right\|^{n-1}_{\mathcal{C}^{s,\sigma}}\left\|(\xi,\eta)\right\|_{\mathcal{P}^{s,\sigma}}. \qquad (1.189)
$$

Proof. Let h,g be as in (1.185). We concentrate on h only, the estimates for g being analogous. Write $h_j=\sum_{k=1}^{N-1}\frac{\partial z_k^n}{\partial\hat{b}_j}\xi_k+\sum_{k=1}^{N-1}\frac{\partial w_k^n}{\partial\hat{b}_j}\eta_k=:h_{j,1}+h_{j,2}$. By (1.180) one gets that

$$
h_{j,1}=\frac{1}{N^{n/2}}\sum_{l=1}^{n}A_j^{n,l}(D\xi,u,\dots,u),\qquad h_{j,2}=\frac{1}{N^{n/2}}\sum_{l=1}^{n}B_j^{n,l}(D\eta,u,\dots,u)
$$

where D is defined in (1.157), the multilinear map $A_j^{n,l}$ is defined by

$$
A_j^{n,l}(h,u,\dots,u)=\sum_{(\mathbf{i},\boldsymbol{\iota})\in\Delta^n}\mathcal{A}_j^{n,l}(\mathbf{i},\boldsymbol{\iota})u_{i_1,\iota_1}\dots h_{i_l}\dots u_{i_n,\iota_n},
$$

$B_j^{n,l}$ is defined analogously but with kernel $\mathcal{B}_j^{n,l}(\mathbf{i},\boldsymbol{\iota})$, and finally $\mathcal{A}_j^{n,l}$ and $\mathcal{B}_j^{n,l}$ are defined for $1\leq j\leq\lfloor N/2\rfloor$ by

$$
\begin{aligned}
\mathcal{A}_j^{n,l}(\mathbf{i},\boldsymbol{\iota})&:=\mathcal{K}_{i_l}^n\Big((i_1,\dots,i_{l-1},j,i_{l+1},\dots,i_n),\,(\iota_1,\dots,\iota_{l-1},1,\iota_{l+1},\dots,\iota_n)\Big),\\
\mathcal{A}_{N-j}^{n,l}(\mathbf{i},\boldsymbol{\iota})&:=\mathcal{K}_{i_l}^n\Big((i_1,\dots,i_{l-1},-j,i_{l+1},\dots,i_n),\,(\iota_1,\dots,\iota_{l-1},1,\iota_{l+1},\dots,\iota_n)\Big),
\end{aligned}
$$

while $\mathcal{B}_j^{n,l}(\mathbf{i},\boldsymbol{\iota})=\overline{\mathcal{A}_j^{n,l}(-\mathbf{i},\boldsymbol{\iota})}$ and $\mathcal{B}_{N-j}^{n,l}(\mathbf{i},\boldsymbol{\iota})=\overline{\mathcal{A}_{N-j}^{n,l}(-\mathbf{i},\boldsymbol{\iota})}$, see (1.180). By Corollary (1.75) it follows that

$$
\begin{aligned}
&\operatorname{supp}\mathcal{A}_j^{n,l}=\operatorname{supp}\mathcal{B}_{N-j}^{n,l}\equiv\{(\mathbf{i},\boldsymbol{\iota}):\,i_1+\dots+i_{l-1}-i_l+i_{l+1}+\dots+i_n=-j,\,\iota_l=1\}\subseteq\Delta_{-j}^n,\\
&\operatorname{supp}\mathcal{A}_{N-j}^{n,l}=\operatorname{supp}\mathcal{B}_j^{n,l}\equiv\{(\mathbf{i},\boldsymbol{\iota}):\,i_1+\dots+i_{l-1}-i_l+i_{l+1}+\dots+i_n=j,\,\iota_l=1\}\subseteq\Delta_j^n.
\end{aligned}
$$

Proceeding as in the proof of Corollary 1.75, one proves that there exists a constant $R > 0$, independent of N, such that (see [KP10])

$$\max_{1\leq l\leq n}\left(\left\|\mathcal{A}_j^{n,l}\right\|_{\Delta^n_{-j}}, \left\|\mathcal{A}_{N-j}^{n,l}\right\|_{\Delta^n_j}, \left\|\mathcal{B}_j^{n,l}\right\|_{\Delta^n_j}, \left\|\mathcal{B}_{N-j}^{n,l}\right\|_{\Delta^n_{-j}}\right) \leq \frac{R^n N^{2(n-1)}}{\langle j\rangle^2}, \quad \forall n \geq 3 \,. \tag{1.190}$$

Thus $\underline{h}$, defined in (1.185), satisfies

$$|\underline{h_j}| \leq \frac{1}{N^{n/2}}\sum_{l=1}^{n}\left(\underline{A_j^{n,l}}(|D\xi|, |u|, \ldots, |u|) + \underline{B_j^{n,l}}(|D\eta|, |u|, \ldots, |u|)\right),$$

where $\underline{A_j^{n,l}}(h, u, \ldots, u) = \sum_{(\mathbf{i},\boldsymbol{\iota})\in\Delta^n}\left|\underline{\mathcal{A}_j^{n,l}(\mathbf{i},\boldsymbol{\iota})}\right| u_{i_1,\iota_1}\ldots h_{i_l}\ldots u_{i_n,\iota_n}$, and $\underline{B_j^{n,l}}$ is defined in analogous way. Then, using (1.190) and arguing as in the proof of Lemma 1.76, one proves the estimate

$$\begin{aligned}\frac{1}{N}\sum_{j=0}^{N-1}[j]_N^{2(s+2)}e^{2\sigma[j]_N}|\underline{h_j}|^2 &\leq R^n N^{4n-5}\left\|(b,a)\right\|^{2(n-1)}_{\mathcal{C}^{s,\sigma}}\left(\frac{1}{N}\sum_{l=1}^{N-1}[l]_N^{2s}e^{2\sigma[l]_N}D_l^2(|\xi_l|^2+|\eta_l|^2)\right)\\ &\leq R^n N^{4n-2}\left\|(b,a)\right\|^{2(n-1)}_{\mathcal{C}^{s,\sigma}}\left\|(\xi,\eta)\right\|^2_{\mathcal{P}^{s-1,\sigma}} \;,\end{aligned}$$

where in the last inequality we used that $D_l^2 \leq \frac{N^3}{[l]_N^2}\omega\left(\frac{l}{N}\right)$. One verifies that $\underline{g}$ satisfies the same inequality, thus estimate (1.189) follows. $\square$

We can finally prove property $(Z4)$. Let $s \geq 0$, $\sigma \geq 0$ be fixed. By Lemma 1.76, 1.77 and 1.78, there exists $C_1, C_2, \epsilon_* > 0$, independent of N, such that for every $0 < \epsilon \leq \epsilon_*$ it holds that

$$\begin{aligned}\sup_{\|(b,a)\|_{\mathcal{C}^{s,\sigma}}\leq\epsilon/N^2}\left\|\underline{Z^0}(b,a)\right\|_{\mathcal{P}^{s+1,\sigma}} &\leq \sum_{n\geq 2}\sup_{\|(b,a)\|_{\mathcal{C}^{s,\sigma}}\leq\epsilon/N^2}\left\|\underline{Z^n}(b,a)\right\|_{\mathcal{P}^{s+1,\sigma}}\\ &\leq \sum_{n\geq 2}R^n N^{2(n-1)}\frac{\epsilon^n}{N^{2n}} \leq \frac{C_1\epsilon^2}{N^2},\\ \sup_{\|(b,a)\|_{\mathcal{C}^{s,\sigma}}\leq\epsilon/N^2}\left\|\underline{dZ^0}(b,a)^*\right\|_{\mathcal{L}(\mathcal{P}^{s,\sigma},\,\mathcal{C}^{s+2,\sigma})} &\leq \sum_{n\geq 2}\sup_{\|(b,a)\|_{\mathcal{C}^{s,\sigma}}\leq\epsilon/N^2}\left\|\underline{dZ^n}(b,a)^*\right\|_{\mathcal{L}(\mathcal{P}^{s,\sigma},\,\mathcal{C}^{s+2,\sigma})}\\ &\leq \sum_{n\geq 2}R^n N^{2n-1}\frac{\epsilon^{n-1}}{N^{2(n-1)}} \leq C_2 N\epsilon \;.\end{aligned}$$

Chapter 2

One smoothing properties of the KdV flow on $\mathbb{R}$

1 Introduction

In the last decades the problem of a rigorous analysis of the theory of infinite dimensional integrable Hamiltonian systems in 1-space dimension has been widely studied. These systems come up in two setups: (i) on compact intervals (finite volume) and (ii) on infinite intervals (infinite volume). The dynamical behaviour of the systems in the two setups have many similar features, but also distinct ones, mostly due to the different manifestation of dispersion.

The analysis of the finite volume case is now quite well understood. Indeed, Kappeler with collaborators introduced a series of methods in order to construct rigorously Birkhoff coordinates (a cartesian version of action-angle variables) for 1-dimensional integrable Hamiltonian PDE's on $\mathbb{T}$. The program succeeded in many cases, like Korteweg-de Vries (KdV) [KP03], defocusing and focusing Nonlinear Schrödinger (NLS) [GK14, KLTZ09]. In each case considered, it has been proved that there exists a real analytic symplectic diffeomorphism, the *Birkhoff map*, between two scales of Hilbert spaces which conjugate the nonlinear dynamics to a linear one.
An important property of the Birkhoff map Φ of the KdV on $\mathbb{T}$ and its inverse Φ^{-1} is the semi-linearity, i.e., the nonlinear part of Φ respectively Φ^{-1} is 1-smoothing. A local version of this result was first proved by Kuksin and Perelman [KP10] and later extended globally by Kappeler, Schaad and Topalov [KST13]. It plays an important role in the perturbation theory of KdV – see [Kuk10] for randomly perturbed KdV equations and [ET13b] for forced and weakly damped problems. The semi-linearity of Φ and Φ^{-1} can be used to prove 1-smoothing properties of the KdV flow in the periodic setup [KST13].

The analysis of the infinite volume case was developed mostly during the '60-'70 of the last century, starting from the pioneering works of Gardner, Greene, Kruskal and Miura [GGKM67, GGKM74] on the KdV on the line. In these works the authors showed that the KdV can be integrated by a *scattering transform* which maps a function q, decaying sufficiently fast at infinity, into the spectral data of the operator $L(q) := -\partial_x^2 + q$. Later, similar results were obtained by Zakharov and Shabat for the NLS on $\mathbb{R}$ [ZS71], by Ablowitz, Kaup, Newell and Segur for the Sine-Gordon equation [AKNS74], and by Flaschka for the Toda lattice with infinitely many particles [Fla74].

Furthermore, using the spectral data of the corresponding Lax operators, action-angle variables were (formally) constructed for each of the equations above [ZF71, ZM74, McL75b, McL75a]. See also [NMPZ84, FT87, AC91] for monographs about the subject. Despite so much work, the analytic properties of the scattering transform and of the action-angle variables in the infinite volume setup are not yet completely understood. In the present paper we discuss these properties, at least for a special class of potentials.

The aim of this paper is to show that for the KdV on the line, the scattering map is an analytic perturbation of the Fourier transform by a 1-smoothing nonlinear operator. With the applications we have in mind, we choose a setup for the scattering map so that the spaces considered are left invariant under the KdV flow. Recall that the KdV equation on $\mathbb{R}$

$$\begin{cases} \partial_t u(t,x) = -\partial_x^3 u(t,x) - 6u(t,x)\partial_x u(t,x) \ , \\ u(0,x) = q(x) \ , \end{cases} \tag{2.1}$$

is globally in time well-posed in various function spaces such as the Sobolev spaces $H^N \equiv H^N(\mathbb{R},\mathbb{R})$, $N \in \mathbb{Z}_{\geq 2}$ (e.g. [BS75, Kat79, KPV93]), as well as on the weighted spaces $H^{2N} \cap L^2_M$, with integers $N \geq M \geq 1$ [Kat66], endowed with the norm $\|\cdot\|_{H^{2N}} + \|\cdot\|_{L^2_M}$. Here $L^2_M \equiv L^2_M(\mathbb{R},\mathbb{C})$ denotes the space of complex valued L^2-functions satisfying $\|q\|_{L^2_M} := \left(\int_{-\infty}^{\infty}(1+|x|^2)^M|q(x)|^2dx\right)^{\frac{1}{2}} < \infty$.

Introduce for $q \in L^2_M$ with $M \geq 4$ the Schrödinger operator $L(q) := -\partial_x^2 + q$ with domain $H^2_{\mathbb{C}}$, where, for any integer $N \in \mathbb{Z}_{\geq 0}$, $H^N_{\mathbb{C}} := H^N(\mathbb{R},\mathbb{C})$. For $k \in \mathbb{R}$ denote by $f_1(q,x,k)$ and $f_2(q,x,k)$ the Jost solutions, i.e. solutions of $L(q)f = k^2 f$ with asymptotics $f_1(q,x,k) \sim e^{ikx}$, $x \to \infty$, $f_2(q,x,k) \sim e^{-ikx}$, $x \to -\infty$. As $f_i(q,\cdot,k)$, $f_i(q,\cdot,-k)$, $i=1,2$, are linearly independent for $k \in \mathbb{R}\setminus\{0\}$, one can find coefficients $S(q,k)$, $W(q,k)$ such that for $k \in \mathbb{R}\setminus\{0\}$ one has

$$\begin{aligned} f_2(q,x,k) =& \frac{S(q,-k)}{2ik} f_1(q,x,k) + \frac{W(q,k)}{2ik} f_1(q,x,-k) \ , \\ f_1(q,x,k) =& \frac{S(q,k)}{2ik} f_2(q,x,k) + \frac{W(q,k)}{2ik} f_2(q,x,-k) \ . \end{aligned} \tag{2.2}$$

It's easy to verify that the functions $W(q,\cdot)$ and $S(q,\cdot)$ are given by the wronskian identities

$$W(q,k) := [f_2, f_1](q,k) := f_2(q,x,k)\partial_x f_1(q,x,k) - \partial_x f_2(q,x,k) f_1(q,x,k) \ , \tag{2.3}$$

and

$$S(q,k) := [f_1(q,x,k), f_2(q,x,-k)] \, , \tag{2.4}$$

which are independent of $x \in \mathbb{R}$. For $q \in \mathcal{Q}$ the functions $S(q,k)$ and $W(q,k)$ are related to the more often used reflection coefficients $r_\pm(q,k)$ and transmission coefficient $t(q,k)$ by the formulas

$$r_+(q,k) = \frac{S(q,-k)}{W(q,k)}, \quad r_-(q,k) = \frac{S(q,k)}{W(q,k)}, \quad t(q,k) = \frac{2ik}{W(q,k)} \quad \forall\, k \in \mathbb{R}\setminus\{0\} \ . \tag{2.5}$$

It is well known that for q real valued the spectrum of $L(q)$ consists of an absolutely continuous part, given by $[0,\infty)$, and a finite number of eigenvalues referred to as bound states, $-\lambda_n < \cdots < -\lambda_1 < 0$ (possibly none). Introduce the set

$$\mathcal{Q} := \left\{ q : \mathbb{R} \to \mathbb{R} \ , \ q \in L^2_4 : W(q,0) \neq 0, \ q \text{ without bound states} \right\} . \tag{2.6}$$

We remark that the property $W(q,0) \neq 0$ is generic. In the sequel we refer to elements in $\mathcal{Q}$ as generic potentials without bound states. Finally we define

$$\mathcal{Q}^{N,M} := \mathcal{Q} \cap H^N \cap L^2_M, \quad N \in \mathbb{Z}_{\geq 0}, \quad M \in \mathbb{Z}_{\geq 4}.$$

We will see in Lemma 2.25 that for any integers $N \geq 0$, $M \geq 4$, $\mathcal{Q}^{N,M}$ is open in $H^N \cap L^2_M$.

Our main theorem analyzes the properties of the scattering map $q \mapsto S(q,\cdot)$ which is known to linearize the KdV flow [GGKM74]. To formulate our result on the scattering map in more details let $\mathscr{S}$ denote the set of all functions $\sigma : \mathbb{R} \to \mathbb{C}$ satisfying

(S1) $\sigma(-k) = \overline{\sigma(k)}, \quad \forall k \in \mathbb{R};$

(S2) $\sigma(0) > 0$.

For $M \in \mathbb{Z}_{\geq 1}$ define the *real* Banach space

$$H^M_\zeta := \{ f \in H^{M-1}_{\mathbb{C}} : \quad \overline{f(k)} = f(-k), \quad \zeta \partial_k^M f \in L^2 \} \ , \tag{2.7}$$

where $\zeta : \mathbb{R} \to \mathbb{R}$ is an odd monotone C^∞ function with

$$\zeta(k) = k \;\text{ for }\; |k| \leq 1/2 \quad \text{ and } \quad \zeta(k) = 1 \;\text{ for }\; k \geq 1 \ . \tag{2.8}$$

The norm on H^M_ζ is given by

$$\|f\|^2_{H^M_\zeta} := \|f\|^2_{H^{M-1}_{\mathbb{C}}} + \left\|\zeta \partial_k^M f\right\|^2_{L^2} .$$

For any $N, M \in \mathbb{Z}_{\geq 0}$ let

$$\mathscr{S}^{M,N} := \mathscr{S} \cap H^M_\zeta \cap L^2_N \ . \tag{2.9}$$

Different choices of ζ, with ζ satisfying (2.8), lead to the same Hilbert space with equivalent norms. We will see in Lemma 2.26 that for any integers $N \geq 0$, $M \geq 4$, $\mathscr{S}^{M,N}$ is an open subset of $H^M_\zeta \cap L^2_N$. Moreover let $\mathcal{F}_\pm$ be the Fourier transformations defined by $\mathcal{F}_\pm(f) = \int_{-\infty}^{+\infty} e^{\mp 2ikx} f(x)\, dx$. In this setup, the scattering map S has the following properties – see Appendix B for a discussion of the notion of real analytic.

Theorem 2.1. *For any integers $N \geq 0$, $M \geq 4$, the following holds:*

(i) The map

$$S : \mathcal{Q}^{N,M} \to \mathscr{S}^{M,N}, \quad q \mapsto S(q,\cdot)$$

is a real analytic diffeomorphism.

(ii) The maps $A := S - \mathcal{F}_-$ and $B := S^{-1} - \mathcal{F}_-^{-1}$ are 1-smoothing, i.e.

$$A : \mathcal{Q}^{N,M} \to H^M_\zeta \cap L^2_{N+1} \quad \text{and} \quad B : \mathscr{S}^{M,N} \to H^{N+1} \cap L^2_{M-1} \ .$$

Furthermore they are real analytic maps.

As a first application of Theorem 2.1 we prove analytic properties of the action variable for the KdV on the line. For a potential $q \in \mathcal{Q}$, the action-angle variable were formally defined for $k \neq 0$ by Zakharov and Faddeev [ZF71] as the densities

$$I(q,k) := \frac{k}{\pi} \log\left(1 + \frac{|S(q,k)|^2}{4k^2}\right) , \quad \theta(q,k) := \arg\left(S(q,k)\right), \quad k \in \mathbb{R} \setminus \{0\} . \tag{2.10}$$

We can write the action as

$$I(q,k) := -\frac{k}{\pi} \log\left(\frac{4k^2}{4k^2 + S(q,k)S(q,-k)}\right) , \quad k \in \mathbb{R} \setminus \{0\} . \tag{2.11}$$

By Theorem 2.1, $S(q,\cdot) \in \mathscr{S}$, thus property (S2) implies that $\lim_{k\to 0} I(q,k)$ exists and equals 0. Furthermore, by (S1), the action $I(q,\cdot)$ is an odd function in k, and strictly positive for $k > 0$. Thus we will consider just the case $k \in [0,+\infty)$. The properties of $I(q,\cdot)$ for k near 0 and k large are described separately.

Corollary 2.2. *For any integers $N \geq 0$, $M \geq 4$, the maps*

$$\mathcal{Q}^{N,M} \to L^1_{2N+1}([1,+\infty),\mathbb{R}) , \quad q \mapsto I(q,\cdot)|_{[1,\infty)}$$

and

$$\mathcal{Q}^{N,M} \to H^M([0,1],\mathbb{R}) , \quad q \mapsto I(q,\cdot)|_{[0,1]} + \frac{k}{\pi} \ln\left(\frac{4k^2}{4(k^2+1)}\right)$$

are real analytic.

Finally we compare solutions of (2.1) to solutions of the Cauchy problem for the Airy equation on $\mathbb{R}$,

$$\begin{cases} \partial_t v(t,x) = -\partial_x^3 v(t,x) \\ v(0,x) = p(x) \end{cases} \tag{2.12}$$

Being a linear equation with constant coefficients, one sees that the Airy equation is globally in time well-posed on H^N and $H^{2N} \cap L^2_M$, with integers $N \geq M \geq 1$ (see Remark 2.41 below). Denote the flows of (2.12) and (2.1) by $U^t_{Airy}(p) := v(t,\cdot)$ respectively $U^t_{KdV}(q) := u(t,\cdot)$. Our third result is to show that for $q \in H^{2N} \cap L^2_M$ with no bound states and $W(q,0) \neq 0$, the difference $U^t_{KdV}(q) - U^t_{Airy}(q)$ is 1-smoothing, i.e. it takes values in H^{2N+1}. More precisely we prove the following theorem.

Theorem 2.3. *Let N, M be integers with $N \geq 2M \geq 8$. Then the following holds true:*

(i) $\mathcal{Q}^{N,M}$ is invariant under the KdV flow.

(ii) For any $q \in \mathcal{Q}^{N,M}$ the difference $U^t_{KdV}(q) - U^t_{Airy}(q)$ takes values in $H^{N+1} \cap L^2_M$. Moreover the map

$$\mathcal{Q}^{N,M} \times \mathbb{R}_{\geq 0} \to H^{N+1} \cap L^2_M, \qquad (q,t) \mapsto U^t_{KdV}(q) - U^t_{Airy}(q)$$

is continuous and for any fixed t real analytic in q.

Outline of the proof: In Section 2 we study analytic properties of the Jost functions $f_j(q,x,k)$, $j = 1,2$, in appropriate Banach spaces. We use these results in Section 3 to prove the direct scattering part of Theorem 2.1. The inverse scattering part of Theorem 2.1 is proved in Section 4. Finally in Section 5 we prove Corollary 2.2 and Theorem 2.3.

Related works: As we mentioned above, this paper is motivated in part from the study of the 1-smoothing property of the KdV flow in the periodic setup, established recently in [BIT11, ET13a, KST13]. In [KST13] the one smoothing property of the Birkhoff map has been exploited to prove that for $q \in H^N(\mathbb{T},\mathbb{R})$, $N \geq 1$, the difference $U^t_{KdV}(q) - U^t_{Airy}(q)$ is bounded in $H^{N+1}(\mathbb{T},\mathbb{R})$ with a bound which grows linearly in time.

Kappeler and Trubowitz [KT86, KT88] studied analytic properties of the scattering map S between weighted Sobolev spaces. More precisely, define the spaces

$$
\begin{aligned}
H^{n,\alpha} &:= \left\{ f \in L^2 : x^\beta \partial_x^j f \in L^2, 0 \leq j \leq n, 0 \leq \beta \leq \alpha \right\} \ , \\
H^{n,\alpha}_\sharp &:= \left\{ f \in H^{n,\alpha} : x^\beta \partial_x^{n+1} f \in L^2, 1 \leq \beta \leq \alpha \right\} \ .
\end{aligned}
$$

In [KT86], Kappeler and Trubowitz showed that the map $q \mapsto S(q,\cdot)$ is a real analytic diffeomorphism from $\mathcal{Q} \cap H^{N,N}$ to $\mathscr{S} \cap H^{N-1,N}_\sharp$, $N \in \mathbb{Z}_{\geq 3}$. They extend their results to potentials with finitely many bound states in [KT88]. Unfortunately, $\mathcal{Q} \cap H^{N,N}$ is not left invariant under the KdV flow.

Results concerning the 1-smoothing property of the inverse scattering map were obtained previously in [Nov96], where it is shown that for a potential q in the space $W^{n,1}(\mathbb{R},\mathbb{R})$ of real-valued functions with weak derivatives up to order n in L^1

$$
q(x) - \frac{1}{\pi} \int_{\mathbb{R}} e^{-2ikx} \chi_c(k) 2ik r_+(q,k) dk \in W^{n+1,1}(\mathbb{R},\mathbb{R}) \ .
$$

Here c is an arbitrary number with $c > \|q\|_{L^1}$ and $\chi_c(k) = 0$ for $|k| \leq c$, $\chi_c(k) = |k| - c$ for $c \leq |k| \leq c+1$, and 1 otherwise. The main difference between the result in [Nov96] and ours concerns the function spaces considered. For the application to the KdV we need to choose function spaces such as $H^N \cap L^2_M$ for which KdV is well posed. To the best of our knowledge it is not known if KdV is well posed in $W^{n,1}(\mathbb{R},\mathbb{R})$. Furthermore in [Nov96] the question of analyticity of the map $q \mapsto r_+(q)$ and its inverse is not addressed.

We remark that Theorem 2.1 treats just the case of regular potentials. In [FHMP09, HMP11] a special class of distributions is considered. In particular the authors study Miura potentials $q \in H^{-1}_{loc}(\mathbb{R},\mathbb{R})$ such that $q = u' + u^2$ for some $u \in L^1(\mathbb{R},\mathbb{R}) \cap L^2(\mathbb{R},\mathbb{R})$, and prove that the map $q \mapsto r_+$ is bijective and locally bi-Lipschitz continuous between appropriate spaces. Finally we point out the work of Zhou [Zho98], in which L^2-Sobolev space bijectivity for the scattering and inverse scattering transforms associated with the ZS-AKNS system are proved.

2 Jost solutions

In this section we assume that the potential q is complex-valued. Often we will assume that $q \in L^2_M$ with $M \in \mathbb{Z}_{\geq 4}$. Consider the normalized Jost functions $m_1(q,x,k) := e^{-ikx} f_1(q,x,k)$ and

$m_2(q,x,k) := e^{ikx} f_2(q,x,k)$ which satisfy the following integral equations

$$m_1(q,x,k) = 1 + \int_x^{+\infty} D_k(t-x)\, q(t)\, m_1(q,t,k) dt \tag{2.13}$$

$$m_2(q,x,k) = 1 + \int_{-\infty}^{x} D_k(x-t)\, q(t)\, m_2(q,t,k) dt \tag{2.14}$$

where $D_k(y) := \int_0^y e^{2iks} ds$.

The purpose of this section is to analyze the solutions of the integral equations (2.13) and (2.14) in spaces needed for our application to KdV. We adapt the corresponding results of [KT86] to these spaces. As (2.13) and (2.14) are analyzed in a similar way we concentrate on (2.13) only. For simplicity we write $m(q,x,k)$ for $m_1(q,x,k)$.

For $1 \leq p \leq \infty$, $M \geq 1$ and $a \in \mathbb{R}$, $1 \leq \alpha < \infty$, $1 \leq \beta \leq \infty$ we introduce the spaces

$$L^p_M := \left\{ f : \mathbb{R} \to \mathbb{C} : \; \langle x \rangle^M f \in L^p \right\} \;, \quad L^\alpha_{x \geq a} L^\beta := \Big\{ f : [a, +\infty) \times \mathbb{R} \to \mathbb{C} : \|f\|_{L^\alpha_{x\geq a} L^\beta} < +\infty \Big\}$$

where $\langle x \rangle := (1+x^2)^{1/2}$, L^p is the standard L^p space, and

$$\|f\|_{L^\alpha_{x\geq a} L^\beta} := \Big(\int_a^{+\infty} \|f(x,\cdot)\|^\alpha_{L^\beta} \; dx \Big)^{1/\alpha}$$

whereas for $\alpha = \infty$, $\|f\|_{L^\infty_{x\geq a} L^\beta} := \sup_{x \geq a} \|f(x,\cdot)\|_{L^\beta}$. We consider also the space $C^0_{x\geq a} L^\beta := C^0\left([a,+\infty), L^\beta\right)$ with $\|f\|_{C^0_{x\geq a} L^\beta} := \sup_{x\geq a} \|f(x,\cdot)\|_{L^\beta} < \infty$. We will use also the space $L^\alpha_{x \leq a} L^\beta$ of functions $f : (-\infty, a] \times \mathbb{R} \to \mathbb{C}$ with finite norm $\|f\|_{L^\alpha_{x\leq a} L^\beta} := \Big(\int_{-\infty}^a \|f(x,\cdot)\|^\alpha_{L^\beta} \; dx \Big)^{1/\alpha}$. Moreover given any Banach spaces X and Y we denote by $\mathcal{L}(X,Y)$ the Banach space of linear bounded operators from X to Y endowed with the operator norm. If $X = Y$, we simply write $\mathcal{L}(X)$.
For the notion of an analytic map between complex Banach spaces we refer to Appendix B.
We begin by stating a well known result about the properties of m.

Theorem 2.4 ([DT79]). *Let $q \in L^1_1$. For each k, $\operatorname{Im} k \geq 0$, the integral equation*

$$m(x,k) = 1 + \int_x^{+\infty} D_k(t-x) q(t) m(t,k) dt \;, \qquad x \in \mathbb{R}$$

has a unique solution $m \in C^2(\mathbb{R}, \mathbb{C})$ which solves the equation $m'' + 2ikm' = q(x)m$ with $m(x,k) \to 1$ as $x \to +\infty$. If in addition q is real valued the function m satisfies the reality condition $\overline{m(q,k)} = m(q,-k)$. Moreover, there exists a constant $K > 0$ which can be chosen uniformly on bounded subsets of L^1_1 such that the following estimates hold for any $x \in \mathbb{R}$

(i) $|m(x,k) - 1| \leq e^{\eta(x)/|k|} \eta(x)/|k|, \quad k \neq 0$;

(ii) $|m(x,k) - 1| \leq K\Big((1 + \max(-x, 0)) \int_x^{+\infty} (1+|t|)|q(t)| dt \Big) / (1 + |k|)$;

(iii) $|m'(x,k)| \leq K_1 \Big(\int_x^{+\infty} (1+|t|)|q(t)| dt \Big) / (1+|k|)$

where $\eta(x) = \int\limits_x^{+\infty} |q(t)|dt$. *For each* x, $m(x,k)$ *is analytic in* $\operatorname{Im} k > 0$ *and continuous in* $\operatorname{Im} k \geq 0$. *In particular, for every* x *fixed,* $k \mapsto m(x,k) - 1 \in H^{2+}$, *where* H^{2+} *is the Hardy space of functions analytic in the upper half plane such that* $\sup_{y>0} \int\limits_{-\infty}^{+\infty} |h(k+iy)|^2 \, dk < \infty$.

Estimates on the Jost functions.

Proposition 2.5. *For any* $q \in L^2_M$ *with* $M \geq 2$, $a \in \mathbb{R}$ *and* $2 \leq \beta \leq +\infty$, *the solution* $m(q)$ *of* (2.13) *satisfies* $m(q) - 1 \in C^0_{x\geq a} L^\beta \cap L^2_{x\geq a} L^2$. *The map* $L^2_M \ni q \mapsto m(q) - 1 \in C^0_{x\geq a} L^\beta \cap L^2_{x\geq a} L^2$ *is analytic. Moreover there exist constants* $C_1, C_2 > 0$, *only dependent on* a, β, *such that*

$$\|m(q) - 1\|_{C^0_{x\geq a} L^\beta} \leq C_1 e^{\|q\|_{L^1_1}} \|q\|_{L^2_1}, \quad \|m(q) - 1\|_{L^2_{x\geq a} L^2} \leq C_2 \|q\|_{L^2_2} \left(1 + \|q\|_{L^2_{3/2}} e^{\|q\|_{L^1_1}}\right). \tag{2.15}$$

Remark 2.6. *In comparison with [KT86], the novelty of Proposition 2.5 consists in the choice of spaces.*

To prove Proposition 2.5 we first need to establish some auxiliary results.

Lemma 2.7. *(i) For any* $q \in L^1_1$, $a \in \mathbb{R}$ *and* $1 \leq \beta \leq +\infty$, *the linear operator*

$$\mathcal{K}(q) : C^0_{x\geq a} L^\beta \to C^0_{x\geq a} L^\beta, \quad f \mapsto \mathcal{K}(q)[f](x,k) := \int\limits_x^{+\infty} D_k(t-x) q(t) f(t,k) dt \tag{2.16}$$

is bounded. Moreover for any $n \geq 1$, *the* n^{th} *composition* $K(q)^n$ *satisfies* $\|\mathcal{K}(q)^n\|_{\mathcal{L}(C^0_{x\geq a} L^\beta)} \leq C^n \|q\|^n_{L^1_1} / n!$ *where* $C > 0$ *is a constant depending only on* a.

(ii) The map $\mathcal{K} : L^1_1 \to \mathcal{L}\left(C^0_{x\geq a} L^\beta\right)$, $q \mapsto \mathcal{K}(q)$, *is linear and bounded, and* $Id - \mathcal{K}$ *is invertible. More precisely,*

$$(Id - \mathcal{K})^{-1} : L^1_1 \to \mathcal{L}\left(C^0_{x\geq a} L^\beta\right), \quad q \mapsto (Id - \mathcal{K}(q))^{-1}$$

is analytic and $\left\|(Id - \mathcal{K})^{-1}\right\|_{\mathcal{L}\left(L^1_1, C^0_{x\geq a} L^\beta\right)} \leq e^{C\|q\|_{L^1_1}}$.

Proof. Let $h \in L^\alpha$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Using $|D_k(t-x)| \leq |t-x|$, one has

$$\begin{aligned} \left| \int\limits_{-\infty}^{+\infty} h(k) \mathcal{K}(q)[f](x,k) dk \right| &\leq \int\limits_x^{+\infty} dt \, |t-x| |q(t)| \, \|f(t,\cdot)\|_{L^\beta} \|h\|_{L^\alpha} \\ &\leq \left(\int\limits_a^{+\infty} |t-a| |q(t)| dt \right) \|f\|_{C^0_{x\geq a} L^\beta} \|h\|_{L^\alpha}, \end{aligned}$$

and hence $\|\mathcal{K}(q)\|_{\mathcal{L}(C^0_{x\geq a} L^\beta)} \leq \int\limits_a^{+\infty} |t-a| |q(t)| dt \leq C \|q\|_{L^1_1}$, where $C > 0$ is a constant depending just on a. To compute the norm of the iteration of the map $\mathcal{K}(q)$ it's enough to proceed as above and exploit the fact that the integration in t is over a simplex, yielding $\|\mathcal{K}(q)^n\|_{C^0_{x\geq a}L^\beta} \leq C^n \|q\|^n_{L^1_1}/n!$ for any $n \geq 1$. Therefore the Neumann series of the operator $\Big(Id - \mathcal{K}(q)\Big)^{-1} = \sum_{n\geq 0} \mathcal{K}(q)^n$ converges absolutely in $\mathcal{L}\left(C^0_{x\geq a}L^\beta\right)$. Since $\mathcal{K}(q)$ is linear and bounded in q, the analyticity and, by item (i), the claimed estimate for $(Id-\mathcal{K})^{-1}$ follow. $\square$

Lemma 2.8. *Let $a \in \mathbb{R}$.*

(i) For any $q \in L^2_{3/2}$, $\mathcal{K}(q)$ defines a bounded linear operator $L^2_{x\geq a}L^2 \to L^2_{x\geq a}L^2$. Moreover the n^{th} composition $K(q)^n$ satisfies

$$\|\mathcal{K}(q)^n\|_{\mathcal{L}(L^2_{x\geq a}L^2)} \leq C^n \|q\|_{L^2_{3/2}} \|q\|^{n-1}_{L^1_1}/(n-1)!$$

where $C > 0$ depends only on a.

(ii) The map $\mathcal{K} : L^2_{3/2} \to \mathcal{L}\left(L^2_{x\geq a}L^2\right), \quad q \mapsto \mathcal{K}(q)$ is linear and bounded; the map

$$(Id-\mathcal{K})^{-1} : L^2_{3/2} \to \mathcal{L}\left(L^2_{x\geq a}L^2\right) \quad q \mapsto (Id-\mathcal{K}(q))^{-1}$$

is analytic and $\left\|(Id-\mathcal{K})^{-1}\right\|_{\mathcal{L}(L^2_{3/2}, L^2_{x\geq a}L^2)} \leq C\left(1 + \|q\|_{L^2_{3/2}} e^{\|q\|_{L^1_1}}\right)$.

Proof. Proceeding as in the proof of the previous lemma, one gets for $x \geq a$ the estimate

$$\|\mathcal{K}(q)[f](x,\cdot)\|_{L^2} \leq \int_x^{+\infty} |t-x||q(t)| \, \|f(t,\cdot)\|_{L^2} \, dt \leq \Big(\int_x^{+\infty} (t-x)^2|q(t)|^2 \, dt\Big)^{1/2} \|f\|_{L^2_{x\geq a}L^2} ,$$

from which it follows that

$$\|\mathcal{K}(q)[f]\|^2_{L^2_{x\geq a}L^2} \leq \left\| \int_x^{+\infty} (t-x)^2|q(t)|^2 \, dt \right\|^{1/2}_{L^1_{x\geq a}} \|f\|_{L^2_{x\geq a}L^2} \leq C \|q\|_{L^2_{3/2}} \|f\|_{L^2_{x\geq a}L^2}$$

proving item (i). To estimate the composition $\mathcal{K}(q)^n$ viewed as an operator on $L^2_{x\geq a}L^2$, remark that

$$\begin{aligned}
\|\mathcal{K}(q)^n[f](x,\cdot)\|_{L^2} &\leq \int_{x\leq t_1\leq \ldots \leq t_n} |t_1-x||q(t_1)| \cdots |t_n - t_{n-1}||q(t_n)| \, \|f(t_n,\cdot)\|_{L^2} \, dt \\
&\leq \int_{x\leq t_1\leq \ldots \leq t_n} |t_1-x||q(t_1)| \cdots |t_{n-1} - t_{n-2}||q(t_{n-1})| \Big(\int_{t_{n-1}}^{+\infty} dt_n \, (t_n - t_{n-1})^2 \, |q(t_n)|^2\Big)^{1/2} \|f\|_{L^2_{x\geq a}L^2} \, dt \\
&\leq \Big(\int_x^{+\infty} (t-x)^2|q(t)|^2 \, dt\Big)^{1/2} \|f\|_{L^2_{x\geq a}L^2} \Big(\int_x^{+\infty} |t-x||q(t)| \, dt\Big)^{n-1} /(n-1)! \, .
\end{aligned}$$

Therefore

$$\|\mathcal{K}(q)^n[f]\|_{L^2_{x\geq a}L^2} \leq \left\| \int_x^{+\infty} (t-x)^2 |q(t)|^2 \, dt \right\|_{L^1_{x\geq a}}^{1/2} \|f\|_{L^2_{x\geq a}L^2} \frac{C^{n-1}\|q\|_{L^1_1}^{n-1}}{(n-1)!}$$
$$\leq \|q\|_{L^2_{3/2}} \|f\|_{L^2_{x\geq a}L^2} C^n \frac{\|q\|_{L^1_1}^{n-1}}{(n-1)!}$$

from which item (i) follows. Item (ii) is then proved as in the previous Lemma. $\square$

Note that for $f \equiv 1$, the expression in (2.16) of $\mathcal{K}(q)[f]$, $\mathcal{K}(q)[1](x,k) = \int\limits_x^{+\infty} D_k(t-x)\, q(t)\, dt$ is well defined.

Lemma 2.9. *For any $2 \leq \beta \leq +\infty$ and $a \in \mathbb{R}$, the map $L^2_2 \ni q \mapsto \mathcal{K}(q)[1] \in C^0_{x\geq a}L^\beta \cap L^2_{x\geq a}L^2$ is analytic. Furthermore*

$$\|\mathcal{K}(q)[1]\|_{C^0_{x\geq a}L^\beta} \leq C_1 \|q\|_{L^2_2}, \quad \|\mathcal{K}(q)[1]\|_{L^2_{x\geq a}L^2} \leq C_2 \|q\|_{L^2_2},$$

where $C_1, C_2 > 0$ are constants depending on a and β.

Proof. Since the map $q \mapsto \mathcal{K}(q)[1]$ is linear in q, it suffices to prove its continuity in q. Moreover, it is enough to prove the result for $\beta = 2$ and $\beta = +\infty$ as the general case then follows by interpolation. For any $k \in \mathbb{R}$, the bound $|D_k(y)| \leq |y|$ shows that the map $k \mapsto D_k(y)$ is in L^∞. Thus

$$\|\mathcal{K}(q)[1](x,\cdot)\|_{L^\infty} \leq \int_x^{+\infty} (t-x)|q(t)| dt \leq \int_a^{+\infty} |t-a||q(t)| \, dt \leq C \|q\|_{L^1_1},$$

where $C > 0$ is a constant depending only on $a \in \mathbb{R}$. The claimed estimate follows by noting that $\|q\|_{L^1_1} \leq C \|q\|_{L^2_2}$.
Using that for $|k| \geq 1$, $|D_k(y)| \leq \frac{1}{|k|}$, one sees that $k \mapsto D_k(y)$ is L^2-integrable. Hence $k \mapsto D_k(t-x)D_{-k}(s-x)$ is integrable. Actually, since the Fourier transform $\mathcal{F}_+(D_k(y))$ in the k-variable of the function $k \mapsto D_k(y)$ is the function $\eta \mapsto \mathbb{1}_{[0,y]}(\eta)$, by Plancherel's Theorem

$$\int_{-\infty}^{\infty} D_k(t-x)\overline{D_k(s-x)} \, dk = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathbb{1}_{[0,t-x]}(\eta) \mathbb{1}_{[0,s-x]}(\eta) \, d\eta = \frac{1}{\pi} \min(t-x, s-x).$$

For any $x \geq a$ one thus has

$$\|\mathcal{K}(q)[1](x,\cdot)\|_{L^2}^2 = \int_{-\infty}^{\infty} \mathcal{K}(q)[1](x,\cdot) \cdot \overline{\mathcal{K}(q)[1]}(x,\cdot) \, dk$$
$$= \iint\limits_{[x,\infty)\times[x,\infty)} dt\, ds\, q(t)\, \overline{q(s)} \int_{-\infty}^{+\infty} D_k(t-x) D_{-k}(s-x) \, dk \;.$$

and hence

$$\|\mathcal{K}(q)[1](x,\cdot)\|_{L^2}^2 \leq \frac{2}{\pi}\int_x^{+\infty}(t-x)|q(t)|\int_t^{+\infty}|q(s)|ds \leq \frac{2}{\pi}\int_a^{+\infty} ds\,|q(s)|\int_a^s |t-a|\,|q(t)|\,dt \leq C\,\|q\|_{L_1^2}^2 \ , \tag{2.17}$$

where the last inequality follows from the Hardy-Littlewood inequality. The continuity in x follows from Lebesgue convergence Theorem.
To prove the second inequality, start from the second term in (2.17) and change the order of integration to obtain

$$\|\mathcal{K}(q)[1]\|_{L^2_{x\geq a}L^2}^2 \leq \left\| \int_x^{+\infty}|t-a||q(t)|\int_t^{+\infty}|q(s)|ds\right\|_{L^1_{x\geq a}} \leq \int_a^{+\infty}|q(s)|\int_a^s (s-a)^2|q(s)|ds \leq C\,\|q\|_{L_1^2}\,\|q\|_{L_2^2}\,.$$

□

Proof of Proposition 2.5. Formally, the solution of equation (2.13) is given by

$$m(q)-1 = \Big(Id-\mathcal{K}(q)\Big)^{-1}\mathcal{K}(q)[1]. \tag{2.18}$$

By Lemma 2.7, 2.8, 2.9 it follows that the r.h.s. of (2.18) is an element of $C^0_{x\geq a}L^\beta \cap L^2_{x\geq a}L^2$, $2\leq\beta\leq\infty$, and analytic as a function of q, since it is the composition of analytic maps. □

Properties of $\partial_k^n m(q,x,k)$ for $1\leq n\leq M-1$. In order to study $\partial_k^n m(q,x,k)$, we deduce from (2.13) an integral equation for $\partial_k^n m(q,x,\cdot)$ and solve it. Recall that for any $M\in\mathbb{Z}_{\geq 0}$, $H^M_{\mathbb{C}}\equiv H^M(\mathbb{R},\mathbb{C})$ denotes the Sobolev space of functions $\{f\in L^2|\ \hat{f}\in L^2_M\}$. The result is summarized in the following

Proposition 2.10. *Fix $M\in\mathbb{Z}_{\geq 4}$ and $a\in\mathbb{R}$. For any integer $1\leq n\leq M-1$ the following holds:*

- *(i) for $q\in L^2_M$ and $x\geq a$ fixed, the function $k\mapsto m(q,x,k)-1$ is in $H^{M-1}_{\mathbb{C}}$;*
- *(ii) the map $L^2_M\ni q\mapsto \partial_k^n m(q)\in C^0_{x\geq a}L^2$ is analytic. Moreover $\|\partial_k^n m(q)\|_{C^0_{x\geq a}L^2}\leq K\,\|q\|_{L^2_M}$, where K can be chosen uniformly on bounded subsets of L^2_M.*

Remark 2.11. *In [CK87b] it is proved that if $q\in L^1_{M-1}$ then for every $x\geq a$ fixed the map $k\mapsto m(q,x,k)$ is in C^{M-2}; note that since $L^2_M\subset L^1_{M-1}$, we obtain the same regularity result by Sobolev embedding theorem.*

To prove Proposition 2.10 we first need to derive some auxiliary results. Assuming that $m(q,x,\cdot)-1$ has appropriate regularity and decay properties, the n^{th} derivative $\partial_k^n m(q,x,k)$ satisfies the following integral equation

$$\partial_k^n m(q,x,k) = \sum_{j=0}^n \binom{n}{j}\int_x^{+\infty}\partial_k^j D_k(t-x)\,q(t)\,\partial_k^{n-j}m(q,t,k)\,dt\ . \tag{2.19}$$

To write (2.19) in a more convenient form introduce for $1 \le j \le n$ and $q \in L^2_{n+1}$ the operators

$$\mathcal{K}_j(q) : C^0_{x\ge a}L^2 \to C^0_{x\ge a}L^2, \quad f \mapsto \mathcal{K}_j(q)[f](x,k) := \int_x^{+\infty} \partial_k^j D_k(t-x)\, q(t)\, f(t,k)\, dt \tag{2.20}$$

leading to

$$\Big(Id - \mathcal{K}(q)\Big)\partial_k^n m(q) = \left(\sum_{j=1}^{n-1} \binom{n}{j} \mathcal{K}_j(q)[\partial_k^{n-j} m(q)] + \mathcal{K}_n(q)[m(q)-1] + \mathcal{K}_n(q)[1]\right). \tag{2.21}$$

In order to prove the claimed properties for $\partial_k^n m(q)$ we must show in particular that the r.h.s. of (2.21) is in $C^0_{x\ge a}L^2$. This is accomplished by the following

Lemma 2.12. *Fix $M \in \mathbb{Z}_{\ge 4}$ and $a \in \mathbb{R}$. Then there exists a constant $C > 0$, depending only on a, M, such that the following holds:*

(i) for any integers $1 \le n \le M-1$

(i1) the map $L^2_M \ni q \mapsto \mathcal{K}_n(q)[1] \in C^0_{x\ge a}L^2$ is analytic, and $\|\mathcal{K}_n(q)[1]\|_{C^0_{x\ge a}L^2} \le C\,\|q\|_{L^2_M}$.

(i2) the map $L^2_M \ni q \mapsto \mathcal{K}_n(q) \in \mathcal{L}\left(L^2_{x\ge a}L^2, C^0_{x\ge a}L^2\right)$ is analytic. Moreover

$$\|\mathcal{K}_n(q)[f]\|_{C^0_{x\ge a}L^2} \le \|q\|_{L^2_M}\, \|f\|_{L^2_{x\ge a}L^2} \ .$$

(ii) For any $1 \le n \le M-2$, the map $L^2_M \ni q \mapsto \mathcal{K}_n(q) \in \mathcal{L}\left(C^0_{x\ge a}L^2\right)$ is analytic. Moreover one has $\|\mathcal{K}_n(q)[f]\|_{C^0_{x\ge a}L^2} \le C\,\|q\|_{L^2_M}\, \|f\|_{C^0_{x\ge a}L^2}$.

(iii) As an application of item (i) and (ii), for any integers $1 \le n \le M-1$ the map $L^2_M \ni q \mapsto \mathcal{K}_n(q)[m(q)-1] \in C^0_{x\ge a}L^2$ is analytic, and

$$\|\mathcal{K}_n(q)[m(q)-1]\|_{C^0_{x\ge a}L^2} \le K_0'\,\|q\|^2_{L^2_M} \ ,$$

where $K_0' > 0$ can be chosen uniformly on bounded subsets of L^2_M.

Proof. First, remark that all the operators $q \mapsto \mathcal{K}_n(q)$ are linear in q, therefore the continuity in q implies the analyticity in q. We begin proving item (i).

$(i1)$ Let $\varphi(x,k) := \int_x^{+\infty} \partial_k^n D_k(t-x)\, q(t)\, dt$ and compute the Fourier transform $\mathcal{F}_+(\varphi(x,\cdot))$ with respect to the k variable for $x \ge a$ fixed, which we denote by $\hat\varphi(x,\xi) \equiv \int_{-\infty}^{\infty} dk\, e^{ik\xi}\varphi(x,k)$. Explicitly

$$\hat\varphi(x,\xi) = \int_x^{+\infty} dt\, q(t) \int_{-\infty}^{+\infty} dk\, e^{2ik\xi}\, \partial_k^n D_k(t-x) = \int_x^{+\infty} q(t)\, \xi^n\, \mathbb{1}_{[0,t-x]}(\xi)\, dt.$$

By Parseval's Theorem $\|\varphi(x,\cdot)\|_{L^2} = \frac{1}{\sqrt{\pi}} \|\hat{\varphi}(x,\cdot)\|_{L^2}$. By changing the order of integration one has

$$\|\hat{\varphi}(x,\cdot)\|_{L^2}^2 = \int_{-\infty}^{+\infty} \hat{\varphi}(x,\xi)\, \overline{\hat{\varphi}(x,\xi)}\, d\xi = \iint_{[x,\infty)\times[x,\infty)} dt\, ds\, q(t)\, \overline{q(s)} \int_{-\infty}^{+\infty} |\xi|^{2n}\, \mathbb{1}_{[0,t-x]}(\xi)\, \mathbb{1}_{[0,s-x]}(\xi) d\xi \leq$$

$$\leq 2 \int_x^{+\infty} dt\, |q(t)|\, |t-x|^{2n+1} \int_t^{+\infty} |q(s)|\, ds \leq \left\|(t-a)^{n+1} q\right\|_{L^2_{t\geq a}} \left\|(t-a)^n \int_t^{+\infty} |q(s)| ds\right\|_{L^2_{t\geq a}}$$

$$\leq C\, \|q\|^2_{L^2_{n+1}},$$

where we used that by $(A3)$ in Appendix A, $\left\|(t-a)^n \int_t^{+\infty} |q(s)|\, ds\right\|_{L^2_{t\geq a}} \leq C\, \|q\|_{L^2_{n+1}}$.

$(i2)$ Let $f \in L^2_{x\geq a} L^2$, and using $|\partial_k^n D_k(t-x)| \leq 2^n |t-x|^{n+1}$ it follows that

$$\|\mathcal{K}_n(q)[f](x,\cdot)\|_{L^2} \leq C \int_x^{+\infty} |q(t)|\, |t-x|^{n+1}\, \|f(t,\cdot)\|_{L^2}\; dt \leq C\, \|q\|_{L^2_{n+1}}\, \|f\|_{L^2_{x\geq a}L^2};$$

by taking the supremum in the x variable one has $\mathcal{K}_n(q) \in \mathcal{L}\left(L^2_{x\geq a}L^2, C^0_{x\geq a}L^2\right)$, where the continuity in x follows by Lebesgue's convergence theorem. The map $q \mapsto \mathcal{K}_n(q)$ is linear and continuous, therefore also analytic.

We prove now item (ii). Let $g \in C^0_{x\geq a}L^2$. From $\|\mathcal{K}_n(q)[g](x,\cdot)\|_{L^2} \leq \int_x^{+\infty} |q(t)|\, |t-x|^{n+1}\, \|g(t,\cdot)\|_{L^2}\; dt$ it follows that

$$\sup_{x\geq a} \|\mathcal{K}_n(q)[g](x,\cdot)\|_{L^2} \leq \|g\|_{C^0_{x\geq a}L^2} \int_a^{+\infty} |q(t)|\, |t-a|^{n+1}\; dt \leq C\, \|g\|_{C^0_{x\geq a}L^2}\, \|q\|_{L^2_{n+2}},$$

which implies the claimed estimate. The analyticity follows from the linearity and continuity of the map $q \mapsto \mathcal{K}_n(q)$.

Finally we prove item (iii). By Proposition 2.5, the map $L^2_{n+1} \ni q \mapsto m(q) - 1 \in L^2_{x\geq a}L^2$ is analytic. By item $(i2)$ above the bilinear map $L^2_{n+1} \times L^2_{x\geq a}L^2 \ni (q,f) \mapsto K_n(q)[f] \in C^0_{x\geq a}L^2$ is analytic; since the composition of analytic maps is analytic, the map $L^2_{n+1} \ni q \mapsto K_n(q)[m(q)-1] \in C^0_{x\geq a}L^2$ is analytic. By $(i2)$ and Proposition 2.5 one has

$$\|\mathcal{K}_n(q)[m(q)-1]\|_{C^0_{x\geq a}L^2} \leq C\, \|q\|_{L^2_{n+1}}\, \|m(q)-1\|_{L^2_{x\geq a}L^2} \leq K_0'\, \|q\|^2_{L^2_{n+1}}$$

where K_0' can be chosen uniformly on bounded subsets of L^2_M. □

Proof of Proposition (2.10). The proof is carried out by a recursive argument in n. We assume that $q \mapsto \partial_k^r m(q)$ is analytic as a map from L^2_M to $C^0_{x\geq a}L^2$ for $0 \leq r \leq n-1$, and prove that $L^2_M \to C^0_{x\geq a}L^2 : q \mapsto \partial_k^n m(q)$ is analytic, provided that $n \leq M-1$. The case $n=0$ is proved in

Proposition 2.5.
We begin by showing that for every $x \geq a$ fixed $k \mapsto \partial_k^{n-1} m(q,x,k)$ is a function in H^1, therefore it has one more (weak) derivative in the k-variable. We use the following characterization of H^1 function [Bre11]:

$$f \in H^1 \text{ iff there exists a constant } C > 0 \text{ such that } \|\tau_h f - f\|_{L^2} \leq C|h|, \quad \forall h \in \mathbb{R}, \tag{2.22}$$

where $(\tau_h f)(k) := f(k+h)$ is the translation operator. Moreover the constant C above can be chosen to be $C = \|\partial_k u\|_{L^2}$. Starting from (2.21) (with $n-1$ instead of n), an easy computation shows that for every $x \geq a$ fixed $(\tau_h)\partial_k^{n-1} m(q) \equiv \partial_k^{n-1} m(q,x,k+h)$ satisfies the integral equation

$$\begin{aligned}
&(Id - \mathcal{K}(q))(\tau_h \partial_k^{n-1} m(q) - \partial_k^{n-1} m(q)) \\
&\qquad = \int_x^{+\infty} (\tau_h \partial_k^{n-1} D_k(t-x) - \partial_k^{n-1} D_k(t-x)) q(t) (m(q,t,k+h) - 1)\, dt \\
&\qquad + \int_x^{+\infty} (\tau_h \partial_k^{n-1} D_k(t-x) - \partial_k^{n-1} D_k(t-x)) q(t)\, dt \\
&\qquad + \int_x^{+\infty} (\partial_k^{n-1} D_k(t-x))\, q(t)\, \left(m(q,t,k+h) - m(q,t,k) \right)\, dt \\
&\qquad + \sum_{j=1}^{n-2} \binom{n-1}{j} \Big(\int_x^{+\infty} (\tau_h \partial_k^j D_k(t-x) - \partial_k^j D_k(t-x)) q(t) \partial_k^{n-1-j} m(q,t,k+h)\, dt \\
&\qquad + \int_x^{+\infty} \partial_k^j D_k(t-x)\, q(t)\, (\tau_h \partial_k^{n-1-j} m(q,t,k) - \partial_k^{n-1-j} m(q,t,k))\, dt \Big) \\
&\qquad + \int_x^{+\infty} (\tau_h D_k(t-x) - D_k(t-x))\, q(t)\, \partial_k^{n-1} m(q,t,k+h)\, dt.
\end{aligned} \tag{2.23}$$

In order to estimate the term in the fourth line on the right hand side of the latter identity, use item $(i1)$ of Lemma 2.12 and the characterization (2.22) of H^1. To estimate all the remaining lines, use the induction hypothesis, the estimates of Lemma 2.12, the fact that the operator norm of $(Id - \mathcal{K}(q))^{-1}$ is bounded uniformly in k and the estimate

$$\left| \tau_h \partial_k^j D_k(t-x) - \partial_k^j D_k(t-x) \right| \leq C|t-x|^{j+2}\, |h|, \quad \forall h \in \mathbb{R},$$

to deduce that for every $n \leq M-1$

$$\left\| \tau_h \partial_k^{n-1} m(q) - \partial_k^{n-1} m(q) \right\|_{L^2} \leq C|h|, \quad \forall h \in \mathbb{R},$$

which is exactly condition (2.22). This shows that $k \mapsto \partial_k^{n-1} m(q,x,k)$ admits a weak derivative in L^2. Formula (2.19) is therefore justified. We prove now that the map $L_M^2 \ni q \mapsto \partial_k^n m(q) \in C^0_{x \geq a} L^2$ is analytic for $1 \leq n \leq M-1$. Indeed equation (2.21) and Lemma 2.12 imply that

$$\|\partial_k^n m(q)\|_{C^0_{x\geq a} L^2} \leq K' \Big(\|q\|_{L_M^2} + \|q\|_{L_M^2}^2 + \sum_{j=1}^{n-1} \|q\|_{L_M^2} \left\| \partial_k^{n-j} m(q) \right\|_{C^0_{x \geq a} L^2} \Big)$$

where K' can be chosen uniformly on bounded subsets of q in L_M^2. Therefore $\partial_k^n m(q) \in C^0_{x\geq a} L^2$ and one gets recursively $\|\partial_k^n m(q)\|_{C^0_{x\geq a} L^2} \leq K \|q\|_{L_M^2}$, where K can be chosen uniformly on bounded subsets of q in L^2_M. The analyticity of the map $q \mapsto \partial_k^n m(q)$ follows by formula (2.21) and the fact that composition of analytic maps is analytic. $\square$

Properties of $k\partial_k^n m(q,x,k)$ for $1 \leq n \leq M$. The analysis of the M^{th} k-derivative of $m(q,x,k)$ requires a separate attention. It turns out that the distributional derivative $\partial_k^M m(q,x,\cdot)$ is not necessarily L^2-integrable near $k=0$ but the product $k\partial_k^M m(q,x,\cdot)$ is. This is due to the fact that $\partial_k^M D_k(x)q(x) \sim x^{M+1}q(x)$ which might not be L^2-integrable. However, by integration by parts, it's easy to see that $k\partial_k^M D_k(x)q(x) \sim x^M q(x) \in L^2$. The main result of this section is the following

Proposition 2.13. *Fix $M \in \mathbb{Z}_{\geq 4}$ and $a \in \mathbb{R}$. Then for every integer $1 \leq n \leq M$ the following holds:*

(i) for every $q \in L^2_M$ and $x \geq a$ fixed, the function $k \mapsto k\partial_k^n m(q,x,k)$ is in L^2;

(ii) the map $L^2_M \ni q \mapsto k\partial_k^n m(q) \in C^0_{x\geq a}L^2$ is analytic. Moreover $\|k\partial_k^n m\|_{C^0_{x\geq a}L^2} \leq K_1 \|q\|_{L^2_M}$ where K_1 can be chosen uniformly on bounded subsets of L^2_M.

Formally, multiplying equation (2.19) by k, the function $k\partial_k^n m(q)$ solves

$$(Id - \mathcal{K}(q))\,(k\partial_k^n m(q)) = \left(\sum_{j=1}^{n-1}\binom{n}{j}\tilde{\mathcal{K}}_j(q)[\partial_k^{n-j}m(q)] + \tilde{\mathcal{K}}_n(q)[m(q)-1] + \tilde{\mathcal{K}}_n(q)[1]\right) \tag{2.24}$$

where we have introduced for $0 \leq j \leq M$ and $q \in L^2_M$ the operators

$$\tilde{\mathcal{K}}_j(q) : C^0_{x\geq a}L^2 \to C^0_{x\geq a}L^2, \quad f \mapsto \tilde{\mathcal{K}}_j(q)[f](x,k) := \int_x^{+\infty} k\partial_k^j D_k(t-x)\,q(t)\,f(t,k)\,dt. \tag{2.25}$$

We begin by proving that each term of the r.h.s. of (2.25) is well defined and analytic as a function of q. The following lemma is analogous to Lemma 2.12:

Lemma 2.14. *Fix $M \in \mathbb{Z}_{\geq 4}$ and $a \in \mathbb{R}$. There exists a constant $C > 0$ such that the following holds:*

(i) for any integers $1 \leq n \leq M$

(i1) the map $L^2_M \ni q \mapsto \tilde{\mathcal{K}}_n(q)[1] \in C^0_{x\geq a}L^2$ is analytic, and $\left\|\tilde{\mathcal{K}}_n(q)[1]\right\|_{C^0_{x\geq a}L^2} \leq C\|q\|_{L^2_M}$;

(i2) the map $L^2_M \ni q \mapsto \tilde{\mathcal{K}}_n(q) \in \mathcal{L}\left(L^2_{x\geq a}L^2,\, C^0_{x\geq a}L^2\right)$ is analytic. Moreover

$$\left\|\tilde{\mathcal{K}}_n(q)[f]\right\|_{C^0_{x\geq a}L^2} \leq C\|q\|_{L^2_M}\|f\|_{L^2_{x\geq a}L^2} \ ;$$

(ii) for any $1 \leq j \leq M-1$ the map $L^2_M \ni q \mapsto \tilde{\mathcal{K}}_j(q) \in \mathcal{L}\left(C^0_{x\geq a}L^2\right)$ is analytic, and

$$\left\|\tilde{\mathcal{K}}_j(q)[f]\right\|_{C^0_{x\geq a}L^2} \leq C\|q\|_{L^2_M}\|f\|_{C^0_{x\geq a}L^2} \ .$$

(iii) As an application of item (i) and (ii) we get

(iii1) for any $1 \leq n \leq M$, the map $L^2_M \ni q \mapsto \tilde{\mathcal{K}}_n(q)[m(q)-1] \in C^0_{x\geq a}L^2$ is analytic with

$$\left\| \tilde{\mathcal{K}}_n(q)[m(q)-1] \right\|_{C^0_{x\geq a}L^2} \leq K_1' \left\| q \right\|^2_{L^2_M}, \tag{2.26}$$

where K_1' can be chosen uniformly on bounded subsets of L^2_M;

(iii2) for any $1 \leq j \leq n-1$, the map $L^2_M \ni q \mapsto \tilde{\mathcal{K}}_j(q)[\partial_k^{n-j}m(q)] \in C^0_{x\geq a}L^2$ is analytic with

$$\left\| \tilde{\mathcal{K}}_j(q)[\partial_k^{n-j}m(q)] \right\|_{C^0_{x\geq a}L^2} \leq K_2' \left\| q \right\|^2_{L^2_M}, \tag{2.27}$$

where K_2' can be chosen uniformly on bounded subsets of L^2_M.

Proof. (i) Since the maps $q \mapsto \tilde{\mathcal{K}}_n(q)$, $0 \leq n \leq M$, are linear, it is enough to prove that these maps are continuous.

$(i1)$ Introduce $\varphi(x,k) := \int_x^{+\infty} k\partial_k^n D_k(t-x)\, q(t)\, dt$. The Fourier transform $\mathcal{F}_+(\varphi(x,\cdot)$ of φ with respect to the k-variable is given by $\mathcal{F}_+(\varphi(x,\cdot)) \equiv \hat{\varphi}(x,\xi)$, where

$$\hat{\varphi}(x,\xi) = \int_x^{+\infty} dt\, q(t) \int_{-\infty}^{+\infty} dk\, e^{-2ik\xi}\, k\partial_k^n D_k(t-x) = -(2i)^{n-1} \int_x^{+\infty} dt\, q(t)\, \partial_\xi(\xi^n\, \mathbb{1}_{[0,t-x]}(\xi)),$$

where $\partial_\xi(\xi^n \mathbb{1}_{[0,t-x]}(\xi))$ is to be understood in the distributional sense. By Parseval's Theorem $\|\varphi(x,\cdot)\|_{L^2} = \frac{1}{\sqrt{\pi}} \|\hat{\varphi}(x,\cdot)\|_{L^2}$. Let C_0^∞ be the space of smooth, compactly supported functions. Since

$$\|\hat{\varphi}(x,\cdot)\|_{L^2_\xi} = \sup_{\substack{\chi \in C_0^\infty \\ \|\chi\|_{L^2} \leq 1}} \left| \int_{-\infty}^{\infty} \chi(\xi)\, \hat{\varphi}(x,\xi)\, d\xi \right|,$$

one computes

$$\begin{aligned}
\left| \int_{-\infty}^{\infty} \chi(\xi)\, \hat{\varphi}(x,\xi)\, d\xi \right| &= \left| \int_x^{+\infty} dt\; q(t) \int_{-\infty}^{\infty} \chi(\xi)\, \partial_\xi \left(\xi^n \mathbb{1}_{[0,t-x]}(\xi) \right)\, d\xi \right| = \left| \int_x^{+\infty} dt\; q(t) \int_0^{t-x} d\xi\; \xi^n \partial_\xi \chi(\xi) \right| \\
&\leq \left| \int_x^{+\infty} dt\; q(t)\chi(t-x)(t-x)^n \right| + n \left| \int_x^{+\infty} dt\; q(t) \int_0^{t-x} d\xi\; \chi(\xi)\xi^{n-1} \right| \\
&\leq \|q\|_{L^2_M} \|\chi\|_{L^2} + n \left| \int_x^{+\infty} dt\; |q(t)||t-x|^{n-1} \int_0^{t-x} d\xi\, |\chi(\xi)| \right| \\
&\leq \|q\|_{L^2_M} \|\chi\|_{L^2} + n \left| \int_x^{+\infty} dt\; |q(t)||t-x|^n \frac{\int_0^{t-x} d\xi\, |\chi(\xi)|}{|t-x|} \right| \leq C \|q\|_{L^2_M} \|\chi\|_{L^2}
\end{aligned}$$

where the last inequality follows from Cauchy-Schwartz and Hardy inequality, and $C > 0$ is a constant depending on a and M.

$(i2)$ As $|k\partial_k^n D_k(t-x)| \leq 2^n |t-x|^n$ by integration by parts, it follows that for some constant $C > 0$ depending only on a and M,

$$\left\| \tilde{\mathcal{K}}_n(q)[f](x,\cdot) \right\|_{L^2} \leq C \int_x^{+\infty} |t-x|^n \, |q(t)| \, \|f(t,\cdot)\|_{L^2} \; dt \leq C \, \|q\|_{L^2_M} \, \|f\|_{L^2_{x\geq a}L^2} \ .$$

Now take the supremum over $x \geq a$ in the expression above and use Lebesgue's dominated convergence theorem to prove item $(i2)$.

(ii) The claim follows by the estimate

$$\left\| \tilde{\mathcal{K}}_j(q)[f](x,\cdot) \right\|_{L^2} \leq C \int_x^{+\infty} |t-x|^j \, |q(t)| \, \|f(t,\cdot)\|_{L^2} \; dt \leq C \, \|q\|_{L^1_j} \, \|f\|_{C^0_{x\geq a}L^2}$$

and the remark that $\|q\|_{L^1_j} \leq C \, \|q\|_{L^2_M}$ for $0 \leq j \leq M-1$.

(iii) By Propositions 2.5 and 2.10 the maps $L^2_M \ni q \mapsto m(q) - 1 \in C^0_{x\geq a}L^2 \cap L^2_{x\geq a}L^2$ and $L^2_M \ni q \mapsto \partial_k^{n-j} m(q) \in C^0_{x\geq a}L^2$ are analytic; by item (ii) for any $1 \leq n \leq M-1$, the bilinear map $(q,f) \mapsto \tilde{\mathcal{K}}_n(q)[f]$ is analytic from $L^2_M \times C^0_{x\geq a}L^2$ to $C^0_{x\geq a}L^2$. Since the composition of two analytic maps is again analytic, item (iii) follows. Moreover $\tilde{\mathcal{K}}_n(q)[m(q)-1]$, $\tilde{\mathcal{K}}_j(q)[\partial_k^{n-j} m(q)] \in C^0_{x\geq a}L^2$ since $m(q,x,k)$ and $\partial_k^n m(q,x,k)$ are continuous in the x-variable. The estimate (2.26) follows from item (ii) and Proposition 2.5, 2.10.

□

Proof of Proposition 2.13. One proceeds in the same way as in the proof of Proposition 2.10. Given any $1 \leq n \leq M$, we assume that $q \mapsto k\partial_k^r m(q)$ is analytic as a map from L^2_M to $C^0_{x\geq a}L^2$ for $1 \leq r \leq n-1$, and deduce that $q \mapsto k\partial_k^n m(q)$ is analytic as a map from L^2_M to $C^0_{x\geq a}L^2$ and satisfies equation (2.24) (with r instead of n).
We begin by showing that for every $x \geq a$ fixed, $k \mapsto k\partial_k^{n-1} m(q,x,k)$ is a function in H^1. Our argument uses again the characterization (2.22) of H^1. Arguing as for the derivation of (2.23) one

gets the integral equation

$$
\begin{aligned}
&(Id-\mathcal{K}(q))(\tau_h(k\partial_k^{n-1}m(q))-k\partial_k^{n-1}m(q))=\\
&\qquad=\int_x^{+\infty}\big(\tau_h(k\partial_k^{n-1}D_k(t-x))-k\partial_k^{n-1}D_k(t-x)\big)\,q(t)(m(q,t,k+h)-1)\,dt\\
&\qquad+\int_x^{+\infty}\big(\tau_h(k\partial_k^{n-1}D_k(t-x))-k\partial_k^{n-1}D_k(t-x)\big)\,q(t)\,dt\\
&\qquad+\int_x^{+\infty}(k\partial_k^{n-1}D_k(t-x))q(t)\,(m(q,t,k+h)-m(q,t,k))\;dt\\
&\qquad+\sum_{j=1}^{n-2}\binom{n-1}{j}\Big(\int_x^{+\infty}\Big(\tau_h(k\partial_k^{j}D_k(t-x))-k\partial_k^{j}D_k(t-x)\Big)\,q(t)\,\partial_k^{n-1-j}m(q,t,k+h)\,dt\\
&\qquad+\int_x^{+\infty}k\partial_k^{j}D_k(t-x)\,q(t)\;\Big(\tau_h\partial_k^{n-1-j}m(q,t,k)-\partial_k^{n-1-j}m(q,t,k)\Big)\;dt\Big)\\
&\qquad+\int_x^{+\infty}(\tau_hD_k(t-x)-D_k(t-x))\;q(t)\,(k+h)\partial_k^{n-1}m(q,t,k+h)\,dt\;.
\end{aligned}
$$

Using the estimates

$$|\tau_hD_k(t-x)-D_k(t-x)|\le C|t-x|^2|h|$$

and

$$\Big|\tau_h(k\partial_k^{j}D_k(t-x))-k\partial_k^{j}D_k(t-x)\Big|\le C|t-x|^{j+1}\,|h|,\quad \forall h\in\mathbb{R}\;,$$

obtained by integration by parts, the characterization (2.22) of H^1, the inductive hypothesis, estimates of Lemma 2.12 and Lemma 2.8 one deduces that for every $n\le M$

$$\big\|\tau_h(k\partial_k^{n-1}m(q))-k\partial_k^{n-1}m(q)\big\|_{L^2}\le C|h|,\quad \forall h\in\mathbb{R}.$$

This shows that $k\mapsto k\partial_k^{n-1}m(q,x,k)$ admits a weak derivative in L^2. Since

$$k\partial_k^{n}m(q,x,k)=\partial_k(k\partial_k^{n-1}m(q,x,k))-\partial_k^{n-1}m(q,x,k)\;,$$

the estimate above and Proposition 2.10 show that $k\mapsto k\partial_k^{n}m(q,x,k)$ is an L^2 function. Formula (2.19) is therefore justified.
The proof of the analyticity of the map $q\mapsto k\partial_k^{n}m(q)$ is analogous to the one of Proposition 2.10 and it is omitted. □

Analysis of $\partial_x m(q,x,k)$. Introduce a odd smooth monotone function $\zeta:\mathbb{R}\to\mathbb{R}$ with $\zeta(k)=k$ for $|k|\le 1/2$ and $\zeta(k)=1$ for $k\ge 1$. We prove the following

Proposition 2.15. *Fix $M\in\mathbb{Z}_{\ge 4}$ and $a\in\mathbb{R}$. Then the following holds:*

(i) for any integer $0 \leq n \leq M-1$, the map $L^2_M \ni q \mapsto \partial_k^n \partial_x m(q) \in C^0_{x\geq a} L^2$ is analytic, and $\|\partial_k^n \partial_x m(q)\|_{C^0_{x\geq a}L^2} \leq K_2 \|q\|_{L^2_M}$ where K_2 can be chosen uniformly on bounded subsets of L^2_M.

(ii) the map $L^2_M \ni q \mapsto \zeta \partial_k^M \partial_x m(q) \in C^0_{x\geq a} L^2$ is analytic, and $\left\|\zeta \partial_k^M \partial_x m(q)\right\|_{C^0_{x\geq a}L^2} \leq K_3 \|q\|_{L^2_M}$ where K_3 can be chosen uniformly on bounded subsets of L^2_M.

The integral equation for $\partial_x m(q,x,k)$ is obtained by taking the derivative in the x-variable of (2.13):

$$\partial_x m(q,x,k) = -\int_x^{+\infty} e^{2ik(t-x)}\, q(t)\, m(q,t,k)\, dt. \tag{2.28}$$

Taking the derivative with respect to the k-variable one obtains, for $0 \leq n \leq M-1$,

$$\partial_k^n \partial_x m(q,x,k) = -\sum_{j=0}^{n} \binom{n}{j} \int_x^{+\infty} e^{2ik(t-x)}\, (2i(t-x))^j\, q(t)\, \partial_k^{n-j} m(q,t,k)\, dt. \tag{2.29}$$

For $0 \leq j \leq M$ introduce the integral operators

$$\mathcal{G}_j(q) : C^0_{x\geq a}L^2 \to C^0_{x\geq a}L^2, \quad q \mapsto \mathcal{G}_j(q)[f](x,k) := -\int_x^{+\infty} e^{2ik(t-x)}\, (2i(t-x))^j\, q(t)\, f(t,k)\, dt \tag{2.30}$$

and rewrite (2.29) in the more compact form

$$\partial_k^n \partial_x m(q) = \sum_{j=0}^{n-1} \binom{n}{j} \mathcal{G}_j(q)[\partial_k^{n-j} m(q)] + \mathcal{G}_n(q)[m(q)-1] + \mathcal{G}_n(q)[1]. \tag{2.31}$$

Proposition 2.15 (i) follows from Lemma 2.16 below.

The M^{th} derivative requires a separate treatment, as $\partial_k^M m$ might not be well defined at $k=0$. Indeed for $n = M$ the integral $\int_x^{+\infty} e^{2ik(t-x)}\, q(t)\, \partial_k^M m(q,t,k)\, dt$ in (2.29) might not be well defined near $k=0$ since we only know that $k\partial_k^M m(q,x,\cdot) \in L^2$. To deal with this issue we use the function ζ described above. Multiplying (2.31) with $n = M$ by ζ we formally obtain

$$\zeta \partial_k^M \partial_x m(q) = \sum_{j=1}^{M-1} \binom{M}{j} \zeta\, \mathcal{G}_j(q)[\partial_k^{M-j} m(q)] + \zeta\, \mathcal{G}_M(q)[m(q)-1] + \zeta\, \mathcal{G}_M(q)[1] + \mathcal{G}_0(q)[\zeta \partial_k^M m(q)].$$

Proposition 2.15 (ii) follows from item (iii) of Lemma 2.16 and the fact that $\zeta \in L^\infty$:

Lemma 2.16. *Fix $M \in \mathbb{Z}_{\geq 4}$ and $a \in \mathbb{R}$. There exists a constant $C > 0$ such that*

(i) for any integer $0 \leq n \leq M$ the following holds:

(i1) the map $L^2_M \ni q \mapsto \mathcal{G}_n(q)[1] \in C^0_{x\geq a}L^2$ is analytic. Moreover $\|\mathcal{G}_n(q)[1]\|_{C^0_{x\geq a}L^2} \leq C \|q\|_{L^2_M}$.

(i2) *The map* $L^2_M \ni q \mapsto \mathcal{G}_n(q) \in \mathcal{L}\left(L^2_{x\geq a}L^2, C^0_{x\geq a}L^2\right)$ *is analytic and*

$$\|\mathcal{G}_n(q)[f]\|_{C^0_{x\geq a}L^2} \leq C\,\|q\|_{L^2_M}\,\|f\|_{L^2_{x\geq a}L^2}\ .$$

(ii) *For any* $0 \leq j \leq M-1$*, the map* $L^2_M \ni q \mapsto \mathcal{G}_j(q) \in \mathcal{L}\left(C^0_{x\geq a}L^2\right)$ *is analytic, and*

$$\|\mathcal{G}_j(q)[f]\|_{C^0_{x\geq a}L^2} \leq C\,\|q\|_{L^2_M}\,\|f\|_{C^0_{x\geq a}L^2}\ .$$

(iii) *For any* $1 \leq n \leq M-1$*,* $0 \leq j \leq n-1$ *and* $\zeta : \mathbb{R} \to \mathbb{R}$ *odd smooth monotone function with* $\zeta(k) = k$ *for* $|k| \leq 1/2$ *and* $\zeta(k) = 1$ *for* $k \geq 1$*, the following holds:*

(iii1) *the maps* $L^2_M \ni q \to \mathcal{G}_j(q)[\partial_k^{n-j}m(q)] \in C^0_{x\geq a}L^2$ *and* $L^2_M \ni q \to \mathcal{G}_n(q)[m(q)-1] \in C^0_{x\geq a}L^2$ *are analytic. Moreover*

$$\left\|\mathcal{G}_j(q)[\partial_k^{n-j}m(q)]\right\|_{C^0_{x\geq a}L^2}, \quad \|\mathcal{G}_n(q)[m(q)-1]\|_{C^0_{x\geq a}L^2} \leq K_2'\,\|q\|^2_{L^2_M}\,,$$

where K_2' *can be chosen uniformly on bounded subsets of* L^2_M*.*

(iii2) *The map* $L^2_M \ni q \to \mathcal{G}_0(q)[\zeta\partial_k^M m(q)] \in C^0_{x\geq a}L^2$ *is analytic and* $\left\|\mathcal{G}_0(q)[\zeta\partial_k^M m(q)]\right\|_{C^0_{x\geq a}L^2} \leq K_3'\,\|q\|^2_{L^2_M}$ *where* K_3' *can be chosen uniformly on bounded subsets of* L^2_M*.*

Proof. As before it's enough to prove the continuity in q of the maps considered to conclude that they are analytic.

$(i1)$ For $x \geq a$ and any $0 \leq n \leq M$ one has $\|\mathcal{G}_n(q)[1](x,\cdot)\|^2_{L^2} \leq C \int_x^{+\infty} |t-x|^{2n}|q(t)|^2 dt \leq C\,\|q\|^2_{L^2_M}$.
The claim follows by taking the supremum over $x \geq a$ in the inequality above.

$(i2)$ For $x \geq a$ and $0 \leq n \leq M$ one has the bound $\|\mathcal{G}_n(q)[f]\|_{C^0_{x\geq a}L^2} \leq C\,\|q\|_{L^2_n}\,\|f\|_{L^2_{x\geq a}L^2}$, which implies the claimed estimate.

(ii) For $x \geq a$ and $0 \leq j \leq M-1$ one has the bound

$$\|\mathcal{G}_j(q)[f]\|_{C^0_{x\geq a}L^2} \leq C\,\|q\|_{L^1_{M-1}}\,\|f\|_{C^0_{x\geq a}L^2} \leq C\,\|q\|_{L^2_M}\,\|f\|_{C^0_{x\geq a}L^2}\ .$$

$(iii1)$ By Proposition 2.10 one has that for any $1 \leq n \leq M-1$ and $0 \leq j \leq n-1$ the map $L^2_M \ni q \mapsto \partial_k^{n-j}m(q) \in C^0_{x\geq a}L^2$ is analytic. Since composition of analytic maps is again an analytic map, the claim regarding the analyticity follows. The first estimate follows from item (ii). A similar argument can be used to prove the second estimate.

$(iii2)$ By Proposition 2.13, the map $L^2_M \ni q \mapsto \zeta\partial_k^M m(q) \in C^0_{x\geq a}L^2$ is analytic, implying the claim regarding the analyticity. The estimate follows from $\left\|\mathcal{G}_0[\zeta\partial_k^M m(q)]\right\|_{C^0_{x\geq a}L^2} \leq \|q\|_{L^2_M}\,\left\|\zeta\partial_k^M m(q)\right\|_{C^0_{x\geq a}L^2}$.

□

The following corollary follows from the results obtained so far:

Corollary 2.17. *Fix* $M \in \mathbb{Z}_{\geq 4}$*. Then the normalized Jost functions* $m_j(q,x,k)$*,* $j = 1,2$*, satisfy:*

(i) the maps $L^2_M \ni q \mapsto m_j(q,0,\cdot) - 1 \in L^2$ *and* $L^2_M \ni q \mapsto k^\alpha \partial_k^n m_j(q,0,\cdot) \in L^2$ *are analytic for* $1 \leq n \leq M-1$ $[1 \leq n \leq M]$ *if* $\alpha = 0$ $[\alpha = 1]$. *Moreover*

$$\|m_j(q,0,\cdot) - 1\|_{L^2}\,,\ \|k^\alpha \partial_k^n m_j(q,0,\cdot)\|_{L^2} \leq K_1 \|q\|_{L^2_M}\,,$$

where $K_1 > 0$ *can be chosen uniformly on bounded subsets of* L^2_M.

(ii) For $0 \leq n \leq M-1$, *the maps* $L^2_M \ni q \mapsto \partial_k^n \partial_x m_j(q,0,\cdot) \in L^2$ *and* $L^2_M \ni q \mapsto \zeta \partial_k^M \partial_x m_j(q,0,\cdot) \in L^2$ *are analytic. Moreover*

$$\|\partial_k^n \partial_x m_j(q,0,\cdot)\|_{L^2}\,,\ \left\|\zeta \partial_k^M \partial_x m_j(q,0,\cdot)\right\|_{L^2} \leq K_2 \|q\|_{L^2_M}\,,$$

where $K_2 > 0$ *can be chosen uniformly on bounded subsets of* L^2_M.

Proof. The Corollary follows by evaluating formulas (2.13), (2.19), (2.29) at $x = 0$ and using the results of Proposition 2.5, 2.10, 2.13 and 2.15. □

3 One smoothing properties of the scattering map.

The aim of this section is to prove the part of Theorem 2.1 related to the direct problem. To begin, note that by Theorem 2.4, for $q \in L^2_4$ real valued one has $\overline{m_1(q,x,k)} = m_1(q,x,-k)$ and $\overline{m_2(q,x,k)} = m_2(q,x,-k)$; hence

$$\overline{S(q,k)} = S(q,-k)\,, \qquad \overline{W(q,k)} = W(q,-k)\,. \tag{2.32}$$

Moreover one has for any $q \in L^2_4$

$$W(q,k)W(q,-k) = 4k^2 + S(q,k)S(q,-k) \qquad \forall\, k \in \mathbb{R} \setminus \{0\} \tag{2.33}$$

which by continuity holds for $k = 0$ as well. In the case where $q \in \mathcal{Q}$, the latter identity implies that $S(q,0) \neq 0$.
Recall that for $q \in L^2_4$ the Jost solutions $f_1(q,x,k)$ and $f_2(q,x,k)$ satisfy the following integral equations

$$f_1(x,k) = e^{ikx} + \int_x^{+\infty} \frac{\sin k(t-x)}{k} q(t) f_1(t,k)\,dt\,, \tag{2.34}$$

$$f_2(x,k) = e^{-ikx} + \int_{-\infty}^{x} \frac{\sin k(x-t)}{k} q(t) f_2(t,k)\,dt\,. \tag{2.35}$$

Substituting (2.34) and (2.35) into (2.4), (2.3), one verifies that $S(q,k)$, $W(q,k)$ satisfy for $k \in \mathbb{R}$ and $q \in L^2_4$

$$S(q,k) = \int_{-\infty}^{+\infty} e^{ikt} q(t) f_1(q,t,k)\,dt\,, \tag{2.36}$$

$$W(q,k) = 2ik - \int_{-\infty}^{+\infty} e^{-ikt} q(t) f_1(q,t,k)\,dt\,. \tag{2.37}$$

Note that the integrals above are well defined thanks to the estimate in item (ii) of Theorem 2.4. Inserting formula (2.34) into (2.36), one gets that

$$S(q,k) = \mathcal{F}_-(q,k) + O\left(\tfrac{1}{k}\right) \ .$$

The main result of this section is an estimate of

$$A(q,k) := S(q,k) - \mathcal{F}_-(q,k) \ , \tag{2.38}$$

saying that A is 1-smoothing. To formulate the result in a precise way, we need to introduce the following Banach spaces for $M \in \mathbb{Z}_{\geq 1}$

$$\begin{aligned} H^M_* &:= \{ f \in H^{M-1}_{\mathbb{C}} : \quad \overline{f(k)} = f(-k), \quad k\partial_k^M f \in L^2 \} \ , \\ H^M_\zeta &:= \{ f \in H^{M-1}_{\mathbb{C}} : \quad \overline{f(k)} = f(-k), \quad \zeta\partial_k^M f \in L^2 \} \ , \end{aligned}$$

where $\zeta : \mathbb{R} \to \mathbb{R}$ is an odd monotone C^∞ function with $\zeta(k) = k$ for $|k| \leq 1/2$ and $\zeta(k) = 1$ for $k \geq 1$. The norms on H^M_* and H^M_ζ are given by

$$\|f\|^2_{H^M_*} := \|f\|^2_{H^{M-1}_{\mathbb{C}}} + \left\|k\partial_k^M f\right\|^2_{L^2} , \quad \|f\|^2_{H^M_\zeta} := \|f\|^2_{H^{M-1}_{\mathbb{C}}} + \left\|\zeta\partial_k^M f\right\|^2_{L^2} .$$

Note that H^M_* and H^M_ζ are *real* Banach spaces. We will use also the complexification of the Banach spaces above, in which the reality condition $\overline{f(k)} = f(-k)$ is dropped:

$$H^M_{*,\mathbb{C}} := \{ f \in H^{M-1}_{\mathbb{C}} : \quad k\partial_k^M f \in L^2 \}, \qquad H^M_{\zeta,\mathbb{C}} := \{ f \in H^{M-1}_{\mathbb{C}} : \quad \zeta\partial_k^M f \in L^2 \}.$$

Note that for any $M \geq 2$

$$(i)\, H^M_{\mathbb{C}} \subset H^M_{\zeta,\mathbb{C}} \text{ and } H^M_{*,\mathbb{C}} \subset H^M_{\zeta,\mathbb{C}}, \quad (ii)\, fg \in H^M_{\zeta,\mathbb{C}} \qquad \forall\, f \in H^M_{*,\mathbb{C}},\, g \in H^M_{\zeta,\mathbb{C}}. \tag{2.39}$$

We can now state the main theorem of this section. Let $L^2_{M,\mathbb{R}} := \left\{ f \in L^2_M \mid f \text{ real valued } \right\}$.

Theorem 2.18. *Let $N \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 4}$. Then one has:*

(i) The map $q \mapsto A(q,\cdot)$ is analytic as a map from L^2_M to $H^M_{\zeta,\mathbb{C}}$.

(ii) The map $q \mapsto A(q,\cdot)$ is analytic as a map from $H^N_{\mathbb{C}} \cap L^2_4$ to L^2_{N+1}. Moreover

$$\|A(q,\cdot)\|_{L^2_{N+1}} \leq C_A \, \|q\|^2_{H^N_{\mathbb{C}} \cap L^2_4}$$

where the constant $C_A > 0$ can be chosen uniformly on bounded subsets of $H^N_{\mathbb{C}} \cap L^2_4$.

Furthermore for $q \in L^2_{4,\mathbb{R}}$ the map $A(q,\cdot)$ satisfies $\overline{A(q,k)} = A(q,-k)$ for every $k \in \mathbb{R}$. Thus its restrictions $A : L^2_{M,\mathbb{R}} \to H^M_\zeta$ and $A : H^N \cap L^2_4 \to L^2_{N+1}$ are real analytic.

The following corollary follows immediately from identity (2.38), item (ii) of Theorem 2.18 and the properties of the Fourier transform:

Corollary 2.19. *Let $N \in \mathbb{Z}_{\geq 0}$. Then the map $q \mapsto S(q,\cdot)$ is analytic as a map from $H^N_{\mathbb{C}} \cap L^2_4$ to L^2_N. Moreover*

$$\|S(q,\cdot)\|_{L^2_N} \leq C_S \, \|q\|_{H^N_{\mathbb{C}} \cap L^2_4}$$

where the constant $C_S > 0$ can be chosen uniformly on bounded subsets of $H^N_{\mathbb{C}} \cap L^2_4$.

In [KST13], it is shown that in the periodic setup, the Birkhoff map of KdV is 1-smoothing. As the map $q \mapsto S(q,\cdot)$ on the spaces considered can be viewed as a version of the Birkhoff map in the scattering setup of KdV, Theorem 2.18 confirms that a result analogous to the one on the circle holds also on the line.

The proof of Theorem 2.18 consists of several steps. We begin by proving item (i). Since $\mathcal{F}_- : L^2_M \to H^M_{\mathbb{C}}$ is bounded, item (i) will follow from the following proposition:

Proposition 2.20. *Let $M \in \mathbb{Z}_{\geq 4}$, then the map $L^2_M \ni q \mapsto S(q,\cdot) \in H^M_{\zeta,\mathbb{C}}$ is analytic and*

$$\|S(q,\cdot)\|_{H^M_{\zeta,\mathbb{C}}} \leq K_S \, \|q\|_{L^2_M} ,$$

where $K_S > 0$ can be chosen uniformly on bounded subsets of L^2_M.

Proof. Recall that $f_1(q,x,k) = e^{ikx}\, m_1(q,x,k)$ and $f_2(q,x,k) = e^{-ikx}\, m_2(q,x,k)$. The x-independence of $S(q,k)$ implies that

$$S(q,k) = [m_1(q,0,k),\, m_2(q,0,-k)] \ . \tag{2.40}$$

As by Corollary 2.17, $m_j(q,0,\cdot) - 1 \in H^M_{*,\mathbb{C}}$ and $\partial_x m_j(q,0,\cdot) \in H^M_{\zeta,\mathbb{C}}$, $j = 1, 2$, the identity (2.40) yields

$$\begin{aligned} S(q,k) =& (m_1(q,0,k) - 1)\, \partial_x m_2(q,0,-k) - (m_2(q,0,-k) - 1)\, \partial_x m_1(q,0,k) \\ &+ \partial_x m_2(q,0,-k) - \partial_x m_1(q,0,k) \ , \end{aligned}$$

thus $S(q,\cdot) \in H^M_{\zeta,\mathbb{C}}$ by (2.39). The estimate on the norm $\|S(q,\cdot)\|_{H^M_{\zeta,\mathbb{C}}}$ follows by Corollary 2.17. □

Proof of Theorem 2.18 (i). The claim is a direct consequence of Proposition 2.20 and the fact that for any real valued potential q, $\overline{S(q,k)} = S(q,-k)$, $\overline{\mathcal{F}_-(q,k)} = \mathcal{F}_-(q,-k)$ and hence $\overline{A(q,k)} = A(q,-k)$ for any $k \in \mathbb{R}$. □

In order to prove the second item of Theorem 2.18, we expand the map $q \mapsto A(q)$ as a power series of q. More precisely, iterate formula (2.34) and insert the formal expansion obtained in this way in the integral term of (2.36), to get

$$S(q,k) = \mathcal{F}_-(q,k) + \sum_{n\geq 1} \frac{s_n(q,k)}{k^n} \tag{2.41}$$

where, with $dt = dt_0 \cdots dt_n$,

$$s_n(q,k) := \int_{\Delta_{n+1}} e^{ikt_0} q(t_0) \prod_{j=1}^{n} \Big(q(t_j)\, \sin k(t_j - t_{j-1}) \Big) e^{ikt_n}\, dt \tag{2.42}$$

is a polynomial of degree $n+1$ in q (cf Appendix B) and Δ_{n+1} is given by

$$\Delta_{n+1} := \Big\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} : \quad t_0 \leq \cdots \leq t_n \Big\} \, .$$

Since by Proposition 2.20 $S(q,\cdot)$ is in L^2, it remains to control the decay of $A(q,\cdot)$ in k at infinity. Introduce a cut off function χ with $\chi(k) = 0$ for $|k| \leq 1$ and $\chi(k) = 1$ for $|k| > 2$ and consider the series

$$\chi(k) S(q,k) = \chi(k) \mathcal{F}_-(q,k) + \sum_{n\geq 1} \frac{\chi(k) s_n(q,k)}{k^n} . \tag{2.43}$$

Item (ii) of Theorem 2.18 follows once we show that each term $\frac{\chi(k)s_n(q,k)}{k^n}$ of the series is bounded as a map from $H^N_{\mathbb{C}} \cap L^2_4$ into L^2_{N+1} and the series has an infinite radius of convergence in L^2_{N+1}. Indeed the analyticity of the map then follows from general properties of analytic maps in complex Banach spaces, see Remark 2.46.

In order to estimate the terms of the series, we need estimates on the maps $k \mapsto s_n(q,k)$. A first trivial bound is given by

$$\|s_n(q,\cdot)\|_{L^\infty} \le \tfrac{1}{(n+1)!}\, \|q\|^{n+1}_{L^1} . \tag{2.44}$$

However, in order to prove convergence of (2.43), one needs more refined estimates of the norm of $k \mapsto s_n(q,k)$ in L^2_N. In order to derive such estimates, we begin with a preliminary lemma about oscillatory integrals:

Lemma 2.21. *Let $f \in L^1(\mathbb{R}^n,\mathbb{C}) \cap L^2(\mathbb{R}^n,\mathbb{C})$. Let $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$ and*

$$g : \mathbb{R} \to \mathbb{C}, \quad g(k) := \int_{\mathbb{R}^n} e^{ik\alpha\cdot t} f(t)\, dt.$$

Then $g \in L^2$ and for any component $\alpha_i \neq 0$ one has

$$\|g\|_{L^2} \le \int\limits_{\mathbb{R}^{n-1}} \Big(\int\limits_{-\infty}^{+\infty} |f(t)|^2\, dt_i \Big)^{1/2} dt_1 \ldots \widehat{dt_i} \ldots dt_n. \tag{2.45}$$

Proof. The lemma is a variant of Parseval's theorem for the Fourier transform; indeed

$$\|g\|^2_{L^2} = \int_{\mathbb{R}} g(k)\, \overline{g(k)}\, dk = \int\limits_{\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n} e^{ik\alpha\cdot(t-s)} f(t)\overline{f(s)}\, dt\, ds\, dk. \tag{2.46}$$

Integrating first in the k variable and using the distributional identity $\int_{\mathbb{R}} e^{ikx}\, dk = \frac{1}{2\pi}\delta_0$, where δ_0 denotes the Dirac delta function, one gets

$$\|g\|^2_{L^2} = \frac{1}{2\pi} \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} f(t)\, \overline{f(s)}\, \delta(\alpha\cdot(t-s))\, dt\, ds \tag{2.47}$$

Choose an index i such that $\alpha_i \neq 0$; then $\alpha\cdot(t-s) = 0$ implies that $s_i = t_i + c_i/\alpha_i$, where $c_i = \sum_{j\neq i} \alpha_j(t_j - s_j)$. Denoting $d\sigma_i = dt_1 \cdots \widehat{dt_i} \cdots dt_n$ and $d\tilde{\sigma}_i = ds_1 \cdots \widehat{ds_i} \cdots ds_n$, one has, integrating first in the variables s_i and t_i,

$$\begin{aligned}
\|g\|^2_{L^2} &= \frac{1}{2\pi} \int\limits_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} d\sigma_i\, d\tilde{\sigma}_i \int_{\mathbb{R}} f(t_1,\ldots,t_i,\ldots,t_n)\overline{f(s_1,\ldots,t_i + c_i/\alpha_i,\ldots,s_n)} dt_i \\
&\le \int\limits_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} d\sigma_i\, d\tilde{\sigma}_i \Big(\int\limits_{-\infty}^{+\infty} |f(t)|^2\, dt_i \Big)^{1/2} \cdot \Big(\int\limits_{-\infty}^{+\infty} |f(s)|^2\, ds_i \Big)^{1/2} \\
&\le \Big(\int\limits_{\mathbb{R}^{n-1}} d\tilde{\sigma}_i \Big(\int\limits_{-\infty}^{+\infty} |f(s)|^2\, ds_i \Big)^{1/2} \Big)^2
\end{aligned} \tag{2.48}$$

where in the second line we have used the Cauchy-Schwarz inequality and the invariance of the integral $\int\limits_{-\infty}^{+\infty} |f(s_1, \ldots, t_i + c_i/\alpha_i, \ldots, s_n)|^2$ by translation. $\square$

To get bounds on the norm of the polynomials $k \mapsto s_n(q,k)$ in L^2_N it is convenient to study the multilinear maps associated with them:

$$\tilde{s}_n \ : \ \left(H^N_{\mathbb{C}} \cap L^1\right)^{n+1} \to L^2_N \ ,$$

$$(f_0, \cdots, f_n) \mapsto \tilde{s}_n(f_0, \cdots, f_n) := \int_{\Delta_{n+1}} e^{ikt_0} f_0(t_0) \prod_{j=1}^{n} \Big(f_j(t_j) \, \sin(k(t_j - t_{j-1})) \Big) \, e^{ikt_n} \, dt \ .$$

The boundedness of these multilinear maps is given by the following

Lemma 2.22. *For each $n \geq 1$ and $N \in \mathbb{Z}_{\geq 0}$, $\tilde{s}_n : (H^N_{\mathbb{C}} \cap L^1)^{n+1} \to L^2_N$ is bounded. In particular there exist constants $C_{n,N} > 0$ such that*

$$\|\tilde{s}_n(f_0, \ldots, f_n)\|_{L^2_N} \leq C_{n,N} \, \|f_0\|_{H^N_{\mathbb{C}} \cap L^1} \cdots \|f_n\|_{H^N_{\mathbb{C}} \cap L^1} \, . \tag{2.49}$$

For the proof, introduce the operators $I_j : L^1 \to L^\infty$, $j = 1, 2$, defined by

$$I_1(f)(t) := \int\limits_{t}^{+\infty} f(s) \, ds \qquad I_2(f)(t) := \int\limits_{-\infty}^{t} f(s) \, ds. \tag{2.50}$$

It is easy to prove that if $u, v \in H^N_{\mathbb{C}} \cap L^1$, then $u \, I_j(v) \in H^N_{\mathbb{C}} \cap L^1$ and the estimate $\|u \, I_j(v)\|_{H^N_{\mathbb{C}}} \leq \|u\|_{H^N_{\mathbb{C}} \cap L^1} \|v\|_{H^N_{\mathbb{C}} \cap L^1}$ holds for $j = 1, 2$.

Proof of Lemma 2.22. As $\sin x = (e^{ix} - e^{-ix})/2i$ we can write $e^{ikt_0} \Big(\prod_{j=1}^n \sin k(t_j - t_{j-1}) \Big) e^{ikt_n}$ as a sum of complex exponentials. Note that the arguments of the exponentials are obtained by taking all the possible combinations of $\pm$ in the expression $t_0 \pm (t_1 - t_0) \pm \ldots \pm (t_n - t_{n-1}) + t_n$. To handle this combinations, define the set

$$\Lambda_n := \Big\{ \sigma = (\sigma_j)_{1 \leq j \leq n} : \, \sigma_j \in \{\pm 1\} \Big\} \tag{2.51}$$

and introduce

$$\delta_\sigma := \#\{1 \leq j \leq n : \, \sigma_j = -1\}.$$

For any $\sigma \in \Lambda_n$, define $\alpha_\sigma = (\alpha_j)_{0 \leq j \leq n}$ as

$$\alpha_0 = (1 - \sigma_1), \quad \alpha_j = \sigma_j - \sigma_{j+1} \text{ for } 1 \leq j \leq n-1, \quad \alpha_n = 1 + \sigma_n.$$

Note that for any $t = (t_0, \ldots, t_n)$, one has $\alpha_\sigma \cdot t = t_0 + \sum_{j=1}^n \sigma_j (t_j - t_{j-1}) + t_n$.
For every $\sigma \in \Lambda_n$, α_σ satisfies the following properties:

$$(i) \ \alpha_0, \, \alpha_n \in \{2, 0\} \, , \ \alpha_j \in \{0, \pm 2\} \ \forall 1 \leq j \leq n-1; \quad (ii) \ \# \{j | \alpha_j \neq 0\} \ \text{ is odd.} \tag{2.52}$$

Property (i) is obviously true; we prove now (ii) by induction. For $n = 1$, property (ii) is trivial. To prove the induction step $n \rightsquigarrow n+1$, let $\alpha_0 = 1 - \sigma_1, \ldots, \alpha_n = \sigma_n - \sigma_{n+1}$, $\alpha_{n+1} = 1 + \sigma_{n+1}$, and

define $\tilde{\alpha}_n := 1 + \sigma_n \in \{0, 2\}$. By the induction hypothesis the vector $\tilde{\alpha}_\sigma = (\alpha_0, \ldots, \alpha_{n-1}, \tilde{\alpha}_n)$ has an odd number of elements non zero. Case $\tilde{\alpha}_n = 0$: in this case the vector $(\alpha_0, \ldots, \alpha_{n-1})$ has an odd number of non zero elements. Then, since $\alpha_n = \sigma_n - \sigma_{n+1} = \tilde{\alpha}_n - \alpha_{n+1} = -\alpha_{n+1}$, one has that $(\alpha_n, \alpha_{n+1}) \in \{(0,0), (-2,2)\}$. Therefore the vector α_σ has an odd number of non zero elements. Case $\tilde{\alpha}_n = 2$: in this case the vector $(\alpha_0, \ldots, \alpha_{n-1})$ has an even number of non zero elements. As $\alpha_n = 2 - \alpha_{n+1}$, it follows that $(\alpha_n, \alpha_{n+1}) \in \{(2,0), (0,2)\}$. Therefore the vector α_σ has an odd number of non zero elements. This proves (2.52).
As

$$e^{ikt_0}\Big(\prod_{j=1}^{n} \sin k(t_j - t_{j-1})\Big)e^{ikt_n} = \sum_{\sigma \in \Lambda_n} \frac{(-1)^{\delta_\sigma}}{(2i)^n} e^{ik\alpha\cdot t}$$

$\tilde{s}_n$ can be written as a sum of complex exponentials, $\tilde{s}_n(f_0, \ldots, f_n)(k) = \sum_{\sigma\in\Lambda_n} \frac{(-1)^{\delta_\sigma}}{(2i)^n} \tilde{s}_{n,\sigma}(f_0, \ldots, f_n)(k)$ where

$$\tilde{s}_{n,\sigma}(f_0, \ldots, f_n)(k) = \int_{\Delta_{n+1}} e^{ik\alpha\cdot t} f_0(t_0)\cdots f_n(t_n) dt. \tag{2.53}$$

The case $N = 0$ follows directly from Lemma 2.21, since for each $\sigma \in \Lambda_n$ one has by (2.52) that there exists m with $\alpha_m \neq 0$ implying $\|\tilde{s}_{n,\sigma}(f_0, \ldots, f_n)\|_{L^2} \leq C \|f_m\|_{L^2} \prod_{j\neq m} \|f_j\|_{L^1}$, which leads to (2.49).

We now prove by induction that $\tilde{s}_n : (H^N_{\mathbb{C}} \cap L^1)^{n+1} \to L^2_N$ for any $N \geq 1$. We start with $n = 1$. Since we have already proved that $\tilde{s}_1$ is a bounded map from $(L^2 \cap L^1)^2$ to L^2, it is enough to establish the stated decay at ∞. One verifies that

$$\begin{aligned}\tilde{s}_1(f_0, f_1) &= \frac{1}{2i}\int_{-\infty}^{+\infty} e^{2ikt} f_0(t)\, I_1(f_1)(t)\, dt - \frac{1}{2i}\int_{-\infty}^{+\infty} e^{2ikt} f_1(t)\, I_2(f_0)(t)\, dt \\ &= \frac{1}{2i}\mathcal{F}_-(f_0\, I_1(f_1)) - \frac{1}{2i}\mathcal{F}_-(f_1\, I_2(f_0)).\end{aligned}$$

Hence, for each $N \in \mathbb{Z}_{\geq 0}$, $(f_0, f_1) \mapsto \tilde{s}_1(f_0, f_1)$ is bounded as a map from $(H^N_{\mathbb{C}} \cap L^1)^2$ to L^2_N. Moreover

$$\|\tilde{s}_1(f_0, f_1)\|_{L^2_N} \leq C_1 \left(\|f_0\, I_1(f_1)\|_{H^N_{\mathbb{C}}} + \|f_1\, I_2(f_0)\|_{H^N_{\mathbb{C}}}\right) \leq C_{1,N} \|f_0\|_{H^N_{\mathbb{C}}\cap L^1} \|f_1\|_{H^N_{\mathbb{C}}\cap L^1}.$$

We prove the induction step $n \rightsquigarrow n+1$ with $n \geq 1$ for any $N \geq 1$ (the case $N = 0$ has been already treated). The term $\tilde{s}_{n+1}(f_0, \ldots, f_{n+1})$ equals

$$\int_{\Delta_{n+2}} e^{ikt_0} f_0(t_0) \prod_{j=1}^{n}\Big(\sin k(t_j - t_{j-1}) f_j(t_j)\Big) e^{ikt_n} \sin k(t_{n+1} - t_n) e^{ik(t_{n+1}-t_n)} f_{n+1}(t_{n+1})\, dt$$

where we multiplied and divided by the factor e^{ikt_n}. Writing

$$\sin k(t_{n+1} - t_n) = (e^{ik(t_{n+1}-t_n)} - e^{-ik(t_{n+1}-t_n)})/2i\ ,$$

the integral term $\int_{t_n}^{+\infty} e^{ik(t_{n+1}-t_n)} \sin k(t_{n+1} - t_n)\, f_{n+1}(t_{n+1})\, dt_{n+1}$ equals

$$\frac{1}{2i}\int_{t_n}^{+\infty} e^{2ik(t_{n+1}-t_n)} f_{n+1}(t_{n+1})\, dt_{n+1} - \frac{1}{2i} I_1(f_{n+1})(t_n).$$

Since $f_{n+1} \in H^N_{\mathbb{C}}$, for $0 \leq j \leq N-1$ one gets $f^{(j)}_{n+1} \to 0$ when $x \to \infty$, where we wrote $f^{(j)}_{n+1} \equiv \partial^j_k f_{n+1}$. Integrating by parts N-times in the integral expression displayed above one has

$$\frac{1}{2i}\sum_{j=0}^{N-1}\frac{(-1)^{j+1}}{(2ik)^{j+1}}\, f^{(j)}_{n+1}(t_n) + \frac{(-1)^N}{2i(2ik)^N}\int_{t_n}^{+\infty} e^{2ik(t_{n+1}-t_n)} f^{(N)}_{n+1}(t_{n+1})\, dt_{n+1} - \frac{1}{2i} I_1(f_{n+1})(t_n).$$

Inserting the formula above in the expression for $\tilde{s}_{n+1}$, and using the multilinearity of $\tilde{s}_{n+1}$ one gets

$$\tilde{s}_{n+1}(f_0,\ldots,f_{n+1}) = \frac{1}{2i}\sum_{j=0}^{N-1}\frac{(-1)^{j+1}}{(2ik)^{j+1}}\tilde{s}_n(f_0,\ldots,f_n\cdot f^{(j)}_{n+1}) - \frac{1}{2i}\tilde{s}_n(f_0,\ldots,f_n\, I_1(f_{n+1})) \tag{2.54}$$

$$+\frac{(-1)^N}{2i(2ik)^N}\int_{\Delta_{n+2}} e^{ikt_0} f_0(t_0)\prod_{j=1}^{n}\Big(\sin k(t_j-t_{j-1})\, f_j(t_j)\Big)e^{2ikt_{n+1}} f^{(N)}_{n+1}(t_{n+1})\, dt_{n+1}. \tag{2.55}$$

We analyze the first term in the r.h.s. of (2.54). For $0 \leq j \leq N-1$, the function $f^{(j)}_{n+1} \in H^{N-j}_{\mathbb{C}}$ is in L^∞ by the Sobolev embedding theorem. Therefore $f_n \cdot f^{(j)}_{n+1} \in H^{N-j}_{\mathbb{C}} \cap L^1$. By the inductive assumption applied to $N-j$, $\tilde{s}_n(f_0,\ldots,f_n\cdot f^{(j)}_{n+1}) \in L^2_{N-j}$. Therefore $\frac{\chi}{(2ik)^{j+1}}\tilde{s}_n(f_0,\ldots,f_n\cdot f^{(j)}_{n+1}) \in L^2_N$, where χ is chosen as in (2.43). For the second term in (2.54) it is enough to note that $f_n\, I_1(f_{n+1}) \in H^N_{\mathbb{C}}\cap L^1$ and by the inductive assumption it follows that $\tilde{s}_n(f_0,\ldots,f_n\, I_1(f_{n+1})) \in L^2_N$. We are left with (2.55). Due to the factor $(2ik)^N$ in the denominator, we need just to prove that the integral term is L^2 integrable in the k-variable. Since the oscillatory factor $e^{2ikt_{n+1}}$ doesn't get canceled when we express the sine functions with exponentials, we can apply Lemma 2.21, integrating first in L^2 w.r. to the variable t_{n+1}, getting

$$\|\chi\cdot(2.55)\|_{L^2_N} \leq C_{n+1,N}\left\|f^{(N)}_{n+1}\right\|_{L^2}\prod_{j=0}^{n}\|f_j\|_{L^1}.$$

Putting all together, it follows that $\tilde{s}_{n+1}$ is bounded as a map from $(H^N_{\mathbb{C}}\cap L^1)^{n+2}$ to L^2_N for each $N \in \mathbb{Z}_{\geq 0}$ and the estimate (2.49) holds. □

By evaluating the multilinear map $\tilde{s}_n$ on the diagonal, Lemma 2.22 says that for any $N \geq 0$,

$$\|s_n(q,\cdot)\|_{L^2_N} \leq C_{n,N}\,\|q\|^{n+1}_{H^N_{\mathbb{C}}\cap L^1}, \qquad \forall n \geq 1. \tag{2.56}$$

Combining the L^∞ estimate (2.44) with (2.56) we can now prove item (*ii*) of Theorem 2.18:

Proof of Theorem 2.18 (ii). Let χ be the cut off function introduced in (2.43) and set

$$\tilde{A}(q,k) := \sum_{n=1}^{\infty}\frac{\chi(k)s_n(q,k)}{k^n}. \tag{2.57}$$

We now show that for any $\rho > 0$, $\tilde{A}(q,\cdot)$ is an absolutely and uniformly convergent series in L^2_{N+1} for q in $B_\rho(0)$, where $B_\rho(0)$ is the ball in $H^N_{\mathbb{C}}\cap L^1$ with center 0 and radius ρ. By (2.56) the map

$q \mapsto \sum_{n=1}^{N+1} \frac{\chi(k)s_n(q,k)}{k^n}$ is analytic as a map from $H^N_{\mathbb{C}} \cap L^1$ to L^2_{N+1}, being a finite sum of polynomials - cf. Remark 2.46. It remains to estimate the sum

$$\tilde{A}_{N+2}(q,k) := \tilde{A}(q,k) - \sum_{n=1}^{N+1} \frac{\chi(k)s_n(q,k)}{k^n} \ .$$

It is absolutely convergent since by the L^∞ estimate (2.44)

$$\left\| \sum_{n\geq N+2} \frac{\chi s_n(q,\cdot)}{k^n} \right\|_{L^2_{N+1}} \leq \sum_{n\geq N+2} \left\| \frac{\chi(k)}{k^n} \right\|_{L^2_{N+1}} \|s_n(q,\cdot)\|_{L^\infty} \leq C \sum_{n\geq N+2} \frac{\|q\|_{L^1}^{n+1}}{(n+1)!} \tag{2.58}$$

for an absolute constant $C > 0$. Therefore the series in (2.57) converges absolutely and uniformly in $B_\rho(0)$ for every $\rho > 0$. The absolute and uniform convergence implies that for any $N \geq 0$, $q \mapsto \tilde{A}(q,\cdot)$ is analytic as a map from $H^N_{\mathbb{C}} \cap L^1$ to L^2_{N+1}.
It remains to show that identity (2.43) holds, i.e., for every $q \in H^N_{\mathbb{C}} \cap L^1$ one has $\chi A(q,\cdot) = \tilde{A}(q,\cdot)$ in L^2_{N+1}. Indeed, fix $q \in H^N_{\mathbb{C}} \cap L^1$ and choose ρ such that $\|q\|_{H^N_{\mathbb{C}} \cap L^1} \leq \rho$. Iterate formula (2.34) $N' \geq 1$ times and insert the result in (2.36) to get for any $k \in \mathbb{R} \setminus \{0\}$,

$$S(q,k) = \mathcal{F}_-(q,k) + \sum_{n=1}^{N'} \frac{s_n(q,k)}{k^n} + S_{N'+1}(q,k) \ ,$$

where

$$S_{N'+1}(q,k) := \frac{1}{k^{N'+1}} \int_{\Delta_{N'+2}} e^{ikt_0} q(t_0) \prod_{j=1}^{N'+1} \Big(q(t_j) \sin k(t_j - t_{j-1}) \Big) f_1(q, t_{N'+1}, k) \, dt \ .$$

By the definition (2.38) of $A(q,k)$ and the expression of $S_{N'+1}$ displayed above

$$\chi(k)A(q,k) - \sum_{n=1}^{N'} \frac{\chi(k)s_n(q,k)}{k^n} = \chi(k) S_{N'+1}(q,k), \qquad \forall N' \geq 1 \ .$$

Let now $N' \geq N$, then by Theorem 2.4 (ii) there exists a constant K_ρ, which can be chosen uniformly on $B_\rho(0)$ such that

$$\|\chi S_{N'+1}(q,\cdot)\|_{L^2_{N+1}} \leq K_\rho \frac{\|q\|_{L^1_1}^{N'+2}}{(N'+2)!} \leq K_\rho \frac{\rho^{N'+2}}{(N'+2)!} \to 0, \qquad \text{when } N' \to \infty \ ,$$

where for the last inequality we used that $\|q\|_{L^1_1} \leq C \|q\|_{L^2_2}$ for some absolute constant $C > 0$. Since $\lim_{N'\to 0} \sum_{n=1}^{N'} \frac{\chi(k)s_n(q,k)}{k^n} = \tilde{A}(q,k)$ in L^2_{N+1}, it follows that $\chi(k)A(q,k) = \tilde{A}(q,k)$ in L^2_{N+1}. $\square$

For later use we study regularity and decay properties of the map $k \mapsto W(q,k)$. For $q \in L^2_4$ real valued with no bound states it follows that $W(q,k) \neq 0$, $\forall \operatorname{Im} k \geq 0$ by classical results in scattering theory. We define

$$\mathcal{Q}_{\mathbb{C}} := \left\{ q \in L^2_4 : W(q,k) \neq 0, \forall \operatorname{Im} k \geq 0 \right\}, \quad \mathcal{Q}^{N,M}_{\mathbb{C}} := \mathcal{Q}_{\mathbb{C}} \cap H^N_{\mathbb{C}} \cap L^2_M \ . \tag{2.59}$$

We will prove in Lemma 2.25 below that $\mathcal{Q}_{\mathbb{C}}^{N,M}$ is open in $H_{\mathbb{C}}^N \cap L_M^2$. Finally consider the Banach space $W_{\mathbb{C}}^M$ defined for $M \geq 1$ by

$$W_{\mathbb{C}}^M := \{f \in L^\infty : \quad \partial_k f \in H_{\mathbb{C}}^{M-1}\} \ , \tag{2.60}$$

endowed with the norm $\|f\|_{W_{\mathbb{C}}^M}^2 = \|f\|_{L^\infty}^2 + \|\partial_k f\|_{H_{\mathbb{C}}^{M-1}}^2$.
Note that $H_{\mathbb{C}}^M \subseteq W_{\mathbb{C}}^M$ for any $M \geq 1$ and

$$gh \in H_{\zeta,\mathbb{C}}^M \quad \forall\, g \in H_{\zeta,\mathbb{C}}^M,\ \forall\, h \in W_{\mathbb{C}}^M \ . \tag{2.61}$$

The properties of the map W are summarized in the following Proposition:

Proposition 2.23. *For $M \in \mathbb{Z}_{\geq 4}$ the following holds:*

(i) The map $L_M^2 \ni q \mapsto W(q,\cdot) - 2ik + \mathcal{F}_-(q,0) \in H_{\zeta,\mathbb{C}}^M$ is analytic and

$$\|W(q,\cdot) - 2ik + \mathcal{F}_-(q,0)\|_{H_{\zeta,\mathbb{C}}^M} \leq C_W \, \|q\|_{L_M^2} \ ,$$

where the constant $C_W > 0$ can be chosen uniformly on bounded subsets of L_M^2.

(ii) The map $\mathcal{Q}_{\mathbb{C}}^{0,M} \ni q \mapsto 1/W(q,\cdot) \in L^\infty$ is analytic.

(iii) The maps

$$\mathcal{Q}_{\mathbb{C}}^{0,M} \ni q \mapsto \frac{\partial_k^j W(q,\cdot)}{W(q,\cdot)} \in L^2 \quad \textit{for } 0 \leq j \leq M-1 \quad \textit{and} \quad \mathcal{Q}_{\mathbb{C}}^{0,M} \ni q \mapsto \frac{\zeta \partial_k^M W(q,\cdot)}{W(q,\cdot)} \in L^2$$

are analytic. Here ζ is a function as in (2.8).

Proof. The x-independence of the Wronskian function (2.3) implies that

$$W(q,k) = 2ik\, m_2(q,0,k)\, m_1(q,0,k) + [m_2(q,0,k),\ m_1(q,0,k)]. \tag{2.62}$$

Introduce for $j = 1,2$ the functions $\grave{m}_j(q,k) := 2ik\,(m_j(q,0,k) - 1)$. By the integral formula (2.13) one verifies that

$$\begin{aligned} \grave{m}_1(q,k) &= \int_0^{+\infty} \left(e^{2ikt} - 1\right) q(t)\,(m_1(q,t,k) - 1)\,dt + \int_0^{+\infty} e^{2ikt}\, q(t)\,dt - \int_0^{+\infty} q(t)\,dt; \\ \grave{m}_2(q,k) &= \int_{-\infty}^{0} \left(e^{-2ikt} - 1\right) q(t)\,(m_2(q,t,k) - 1)\,dt + \int_{-\infty}^{0} e^{-2ikt}\, q(t)\,dt - \int_{-\infty}^{0} q(t)\,dt. \end{aligned} \tag{2.63}$$

A simple computation using (2.62) shows that $W(q,k) - 2ik + \mathcal{F}_-(q,0) = I + II + III$ where

$$\begin{aligned} &I := \grave{m}_1(q,k) + \grave{m}_2(q,k) + \mathcal{F}_-(q,0), \\ &II := \grave{m}_1(q,k)(m_2(q,0,k) - 1) \quad \text{ and } \quad III := [m_2(q,0,k), m_1(q,0,k)]. \end{aligned} \tag{2.64}$$

We prove now that each of the terms I, II and III displayed above is an element of $H_{\zeta,\mathbb{C}}^M$. We begin by discussing the smoothness of the functions $k \mapsto \grave{m}_j(q,k)$, $j = 1,2$. For any $1 \leq n \leq M$,

$$\partial_k^n \grave{m}_j(q,k) = 2in\, \partial_k^{n-1}(m_j(q,0,k) - 1) + 2ik\, \partial_k^n m_j(q,0,k) \ .$$

Thus by Corollary 2.17 (i), $\grave{m}_j(q,\cdot) \in W^M_{\mathbb{C}}$ and $q \mapsto \grave{m}_j(q,\cdot)$, $j = 1,2$, are analytic as maps from L^2_M to $W^M_{\mathbb{C}}$. Consider first the term III in (2.64). By Corollary 2.17, $\|III(q,\cdot)\|_{H^M_{\zeta,\mathbb{C}}} \leq K_{III}\|q\|_{L^2_M}$, where $K_{III} > 0$ can be chosen uniformly on bounded subsets of L^2_M. Arguing as in the proof of Proposition 2.20, one shows that it is an element of $H^M_{\zeta,\mathbb{C}}$ and it is analytic as a map $L^2_M \to H^M_{\zeta,\mathbb{C}}$. Next consider the term II. Since $\grave{m}_1(q,\cdot)$ is in $W^M_{\mathbb{C}}$ and $m_2(q,0,\cdot) - 1$ is in $H^M_{\zeta,\mathbb{C}}$, it follows by (2.61) that their product is in $H^M_{\zeta,\mathbb{C}}$. It is left to the reader to show that $L^2_M \to H^M_{\zeta,\mathbb{C}}$, $q \mapsto II(q)$ is analytic and furthermore $\|II(q,\cdot)\|_{H^M_{\zeta,\mathbb{C}}} \leq K_{II}\|q\|_{L^2_M}$, where $K_{II} > 0$ can be chosen uniformly on bounded subsets of L^2_M.
Finally let us consider term I. By summing the identities for $\grave{m}_1$ and $\grave{m}_2$ in equation (2.63), one gets that

$$\begin{aligned}\grave{m}_1(q,k) + \grave{m}_2(q,k) + \mathcal{F}_-(q,0) = &\int_0^{+\infty} e^{2ikt}\, q(t)\, m_1(q,t,k)\, dt - \int_0^{+\infty} q(t)\,(m_1(q,t,k) - 1)\, dt \\ &+ \int_{-\infty}^{0} e^{-2ikt}\, q(t)\, m_2(q,t,k) - \int_{-\infty}^{0} q(t)\,(m_2(q,t,k) - 1)\, dt.\end{aligned} \tag{2.65}$$

We study just the first line displayed above, the second being treated analogously. By equation (2.28) one has that $\int_0^{+\infty} e^{2ikt}\, q(t)\, m_1(q,t,k)\, dt = \partial_x m(q,0,k)$, which by Corollary 2.17 is an element of $H^M_{\zeta,\mathbb{C}}$ and analytic as a function $L^2_M \to H^M_{\zeta,\mathbb{C}}$. Furthermore, by Proposition 2.10 and Proposition 2.13 it follows that $k \mapsto \int_0^{+\infty} q(t)\,(m_1(q,t,k) - 1)\, dt$ is an element of $H^M_{\zeta,\mathbb{C}}$ and it is analytic as a function $L^2_M \to H^M_{\zeta,\mathbb{C}}$. This proves item (i). By Corollary 2.17, it follows that $\|I(q,\cdot)\|_{H^M_{\zeta,\mathbb{C}}} \leq K_I\|q\|_{L^2_M}$, where $K_I > 0$ can be chosen uniformly on bounded subsets of L^2_M.
We prove now item (ii). By the definition of $\mathcal{Q}_{\mathbb{C}}$, for $q \in \mathcal{Q}^{0,4}_{\mathbb{C}}$ the function $W(q,k) \neq 0$ for any k with $\operatorname{Im} k \geq 0$. By Proposition 2.20 (ii) and the condition $M \geq 4$, it follows that $W(q,k) = 2ik + L^\infty$; therefore the map $\mathcal{Q}^{0,M}_{\mathbb{C}} \ni q \mapsto 1/W(q) \in L^2$ is analytic.
Item (iii) follows immediately from item (i) and (ii). □

Lemma 2.24. *For any* $q \in \mathcal{Q}^{0,4}$*,* $W(q,0) < 0$.

Proof. Let $q \in \mathcal{Q}^{0,4}$ and $\kappa \geq 0$. By formulas (2.34) and (2.35) with $k = i\kappa$, it follows that $f_j(q,x,i\kappa)$ $(j = 1,2)$ is real valued (recall that q is real valued). By the definition $W(q,i\kappa) = [f_2, f_1]\,(q,i\kappa)$ it follows that for $\kappa \geq 0$, $W(q,i\kappa)$ is real valued. As q is generic, $W(q,i\kappa)$ has no zeroes for $\kappa \geq 0$. Furthermore for large κ we have $W(q,i\kappa) \sim 2i(i\kappa) = -2\kappa$. Thus $W(q,i\kappa) < 0$ for $\kappa \geq 0$. □

We are now able to prove the direct scattering part of Theorem 2.1.

Proof of Theorem 2.1: direct scattering part. Let $N \geq 0$, $M \geq 4$ be fixed integers. First we remark that $S(q,\cdot)$ is an element of $\mathscr{S}^{M,N}$ if $q \in \mathcal{Q}^{N,M}$. By (2.32), $S(q,\cdot)$ satisfies (S1). To see that $S(q,0) > 0$ recall that $S(q,0) = -W(q,0)$, and by Lemma 2.24 $W(q,0) < 0$. Thus $S(q,\cdot)$ satisfies (S2). Finally by Corollary 2.19 and Proposition 2.20 it follows that $S(q,\cdot) \in \mathscr{S}^{M,N}$. The analyticity properties of the map $q \mapsto S(q,\cdot)$ and $q \mapsto A(q,\cdot)$ follow by Corollary 2.19, Proposition 2.20 and

Theorem 2.18. $\square$

We conclude this section with a lemma about the openness of $\mathcal{Q}^{N,M}$ and $\mathscr{S}^{M,N}$.

Lemma 2.25. *For any integers $N \geq 0$, $M \geq 4$, $\mathcal{Q}^{N,M}$ $[\mathcal{Q}^{N,M}_{\mathbb{C}}]$ is open in $H^N \cap L^2_M$ $[H^N_{\mathbb{C}} \cap L^2_M]$.*

Proof. The proof can be found in [KT88]; we sketch it here for the reader's convenience. By a classical result in scattering theory [DT79], $W(q,k)$ admits an analytic extension to the upper plane $\operatorname{Im} k \geq 0$. By definition (2.59) one has $\mathcal{Q}_{\mathbb{C}} = \{q \in L^2_4 : W(q,k) \neq 0 \quad \forall \operatorname{Im} k \geq 0\}$. Using that $(q,k) \mapsto W(q,k)$ is continuous on $L^2_4 \times \mathbb{R}$ and that by Proposition 2.23, $\|W(q,\cdot) - 2ik\|_{L^\infty}$ is bounded locally uniformly in $q \in L^2_4$ one sees that $\mathcal{Q}_{\mathbb{C}}$ is open in L^2_4. The remaining statements follow in a similar fashion. $\square$

Denote by $H^M_{\zeta,\mathbb{C}}$ the complexification of the Banach space H^M_ζ, in which the reality condition $\overline{f(k)} = f(-k)$ is dropped:

$$H^M_{\zeta,\mathbb{C}} := \{f \in H^{M-1}_{\mathbb{C}} : \quad \zeta \partial_k^M f \in L^2\}. \tag{2.66}$$

On $H^M_{\zeta,\mathbb{C}} \cap L^2_N$ with $M \geq 4$, $N \geq 0$, define the linear functional

$$\Gamma_0 : H^M_{\zeta,\mathbb{C}} \cap L^2_N \to \mathbb{C}, \quad h \mapsto h(0).$$

By the Sobolev embedding theorem Γ_0 is a linear analytic map on $H^M_{\zeta,\mathbb{C}} \cap L^2_N$. In view of the definition (2.9), $\mathscr{S}^{M,N} \subseteq H^M_\zeta$. Furthermore denote by $\mathscr{S}^{M,N}_{\mathbb{C}}$ the complexification of $\mathscr{S}^{M,N}$. It consists of functions $\sigma : \mathbb{R} \to \mathbb{C}$ with $\Re(\sigma(0)) > 0$ and $\sigma \in H^M_{\zeta,\mathbb{C}} \cap L^2_N$.
In the following we denote by $C^{n,\gamma}(\mathbb{R},\mathbb{C})$, with $n \in \mathbb{Z}_{\geq 0}$ and $0 < \gamma \leq 1$, the space of complex-valued functions with n continuous derivatives such that the n^{th} derivative is Hölder continuous with exponent γ.

Lemma 2.26. *For any integers $M \geq 4$, $N \geq 0$ the subset $\mathscr{S}^{M,N}$ $[\mathscr{S}^{M,N}_{\mathbb{C}}]$ is open in $H^M_\zeta \cap L^2_N$ $[H^M_{\zeta,\mathbb{C}} \cap L^2_N]$.*

Proof. Clearly $H^4_{\zeta,\mathbb{C}} \subseteq H^3_{\mathbb{C}}$, and by the Sobolev embedding theorem $H^3_{\mathbb{C}} \hookrightarrow C^{2,\gamma}(\mathbb{R},\mathbb{C})$ for any $0 < \gamma < 1/2$. It follows that $\sigma \to \sigma(0)$ is a continuous functional on $H^4_{\zeta,\mathbb{C}}$. In view of the definition of $\mathscr{S}^{M,N}$, the claimed statement follows. $\square$

4 Inverse scattering map

The aim of this section is to prove the inverse scattering part of Theorem 2.1. More precisely we prove the following theorem.

Theorem 2.27. *Let $N \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 4}$ be fixed. Then the scattering map $S : \mathcal{Q}^{N,M} \to \mathscr{S}^{M,N}$ is bijective. Its inverse $S^{-1} : \mathscr{S}^{M,N} \to \mathcal{Q}^{N,M}$ is real analytic.*

The smoothing and analytic properties of $B := S^{-1} - \mathcal{F}_-^{-1}$ claimed in Theorem 2.1 follow now in a straightforward way from Theorem 2.27 and 2.18.

Proof of Theorem 2.1: inverse scattering part. By Theorem 2.27, $S^{-1} : \mathscr{S}^{M,N} \to \mathcal{Q}^{N,M}$ is well defined and real analytic. As by definition $B = S^{-1} - \mathcal{F}_-^{-1}$ and $S = \mathcal{F}_- + A$ one has $B \circ S = Id - \mathcal{F}_-^{-1} \circ S = -\mathcal{F}_-^{-1} \circ A$ or

$$B = -\mathcal{F}_-^{-1} \circ A \circ S^{-1} \ .$$

Hence, by Theorem 2.18 and Theorem 2.27, for any $M \in \mathbb{Z}_{\geq 4}$ and $N \in \mathbb{Z}_{\geq 0}$ the restriction $B : \mathscr{S}^{M,N} \to H^{N+1} \cap L^2_{M-1}$ is a real analytic map. $\square$

The rest of the section is devoted to the proof of Theorem 2.27. By the direct scattering part of Theorem 2.1 proved in Section 3, $S(\mathcal{Q}^{N,M}) \subseteq \mathscr{S}^{M,N}$. Furthermore, the map $S : \mathcal{Q} \to \mathscr{S}$ is 1-1, see [KT86, Section 4]. Thus also its restriction $S|_{\mathcal{Q}^{N,M}} : \mathcal{Q}^{N,M} \to \mathscr{S}^{M,N}$ is 1-1.

Let us denote by $\mathcal{H} : L^2 \to L^2$ the Hilbert transform

$$\mathcal{H}(v)(k) := -\frac{1}{\pi} \, \mathrm{p.v.} \int_{-\infty}^{\infty} \frac{v(k')}{k' - k} dk' \ . \tag{2.67}$$

We collect in Appendix E some well known properties of the Hilbert transform which will be exploited in the following.

In order to prove that $S : \mathcal{Q}^{N,M} \to \mathscr{S}^{M,N}$ is onto, we need some preparation. Following [KT86] define for $\sigma \in \mathscr{S}^{M,N}$,

$$\omega(\sigma, k) := \exp\left(\frac{1}{2} l(\sigma, k) + \frac{i}{2} \mathcal{H}(l(\sigma, \cdot))(k)\right) \ , \quad l(\sigma, k) := \log\left(\frac{4(k^2+1)}{4k^2 + \sigma(k)\sigma(-k)}\right) \ , \quad k \in \mathbb{R} \tag{2.68}$$

and

$$\begin{aligned} \frac{1}{w(\sigma,k)} &:= \frac{\omega(\sigma,k)}{2i(k+i)} \ , \qquad & \tau(\sigma,k) &:= \frac{2ik}{w(\sigma,k)} \ , \\ \rho_+(\sigma,k) &:= \frac{\sigma(-k)}{w(\sigma,k)} \ , \qquad & \rho_-(\sigma,k) &:= \frac{\sigma(k)}{w(\sigma,k)} \ . \end{aligned} \tag{2.69}$$

The aim is to show that $\rho_+(\sigma, \cdot)$, $\rho_-(\sigma, \cdot)$ and $\tau(\sigma, \cdot)$ are the scattering data r_+, r_- and t of a potential $q \in \mathcal{Q}^{N,M}$.

In the next proposition we discuss the properties of the map $\sigma \to l(\sigma, \cdot)$. To this aim we introduce, for $M \in \mathbb{Z}_{\geq 2}$ and ζ as in (2.8), the auxiliary Banach space

$$W_\zeta^M := \{ f \in L^\infty : \overline{f(k)} = f(-k), \quad \partial_k^n f \in L^2 \text{ for } 1 \leq n \leq M-1 \ , \quad \zeta \partial_k^M f \in L^2 \} \tag{2.70}$$

and its complexification

$$W_{\zeta,\mathbb{C}}^M := \{ f \in L^\infty : \quad \partial_k^n f \in L^2 \text{ for } 1 \leq n \leq M-1 \ , \quad \zeta \partial_k^M f \in L^2 \} \ , \tag{2.71}$$

both endowed with the norm $\|f\|^2_{W^M_{\zeta,\mathbb{C}}} := \|f\|^2_{L^\infty} + \|\partial_k f\|^2_{H^{M-2}_{\mathbb{C}}} + \left\|\zeta \partial_k^M f\right\|^2_{L^2}$.

Proposition 2.28. *Let $N \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 4}$ be fixed. The map $\mathscr{S}^{M,N} \to H_\zeta^M$, $\sigma \to l(\sigma, \cdot)$ is real analytic.*

Proof. Denote by

$$h(\sigma, k) := \frac{4(k^2+1)}{4k^2 + \sigma(k)\sigma(-k)} \ .$$

We show that the map $\mathscr{S}^{M,N} \to W^M_\zeta$, $\sigma \to h(\sigma,\cdot)$ is real analytic. First note that the map $\mathscr{S}^{M,N}_{\mathbb{C}} \to L^\infty$, assigning to σ the function $\sigma(k)\sigma(-k)$ is analytic by the Sobolev embedding theorem. For $\sigma \in \mathscr{S}^{M,N}_{\mathbb{C}}$ write $\sigma = \sigma_1 + i\sigma_2$, where $\sigma_1 := \Re\sigma$, $\sigma_2 := \operatorname{Im}\sigma$. Then

$$\Re(\sigma(k)\sigma(-k)) = \sigma_1(k)\sigma_1(-k) - \sigma_2(k)\sigma_2(-k) \ . \tag{2.72}$$

Now fix $\sigma^0 \in \mathscr{S}^{M,N}$ and recall that $\mathscr{S}^{M,N} = \mathscr{S} \cap H^M_\zeta \cap L^2_N$. Remark that $\sigma^0_2 := \operatorname{Im}\sigma^0 = 0$, while $\sigma^0_1 := \Re\sigma^0$ satisfies $\sigma^0_1(k)\sigma^0_1(-k) \geq 0$ and $\sigma^0_1(0)^2 > 0$. Thus, by formula (2.72) and the Sobolev embedding theorem, there exists $V_{\sigma^0} \subset \mathscr{S}^{M,N}_{\mathbb{C}}$ small complex neighborhood of σ^0 and a constant C_{σ^0} such that

$$\Re(4k^2 + \sigma(k)\sigma(-k)) > C_{\sigma^0} \ , \quad \forall \sigma \in V_{\sigma^0} \ .$$

It follows that there exist constants $C_1, C_2 > 0$ such that

$$\Re h(\sigma,k) \geq C_1 \ , \qquad |h(\sigma,k)| \leq C_2 \ , \qquad \forall k \in \mathbb{R},\ \forall \sigma \in V_{\sigma^0} \ , \tag{2.73}$$

implying that the map $V_{\sigma^0} \to L^\infty$, $\sigma \to h(\sigma,\cdot)$ is analytic. In a similar way one proves that $V_{\sigma^0} \to W^M_{\zeta,\mathbb{C}}$, $\sigma \mapsto h(\sigma,\cdot)$ is analytic (we omit the details). If $\overline{\sigma(k)} = \sigma(-k)$, the function $h(\sigma,\cdot)$ is real valued. Thus it follows that $\mathscr{S}^{M,N} \to W^M_\zeta$, $\sigma \to h(\sigma,\cdot)$ is real analytic.

We consider now the map $\sigma \to l(\sigma,\cdot)$. By (2.73), $l(\sigma,k) = \log(h(\sigma,k))$ is well defined for every $k \in \mathbb{R}$. Since the logarithm is a real analytic function on the right half plane, the map $\mathscr{S}^{M,N} \to L^\infty$, $\sigma \to l(\sigma,\cdot)$ is real analytic as well. Furthermore for $|k| > 1$ one finds a constant $C_3 > 0$ such that $|l(\sigma,k)| \leq C_3/|k|^2$, $\forall \sigma \in V_{\sigma^0}$. Thus $\sigma \to l(\sigma,\cdot)$ is real analytic as a map from $\mathscr{S}^{M,N}$ to L^2. One verifies that $\partial_k \log(h(\sigma,\cdot)) = \frac{\partial_k h(\sigma,\cdot)}{h(\sigma,\cdot)}$ is in L^2 and one shows by induction that the map $\mathscr{S}^{M,N} \to H^M_\zeta$, $\sigma \mapsto l(\sigma,\cdot)$ is real analytic. $\square$

In the next proposition we discuss the properties of the map $\sigma \to \omega(\sigma,\cdot)$.

Proposition 2.29. *Let $N \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 4}$ be fixed. The map $\mathscr{S}^{M,N} \to W^M_\zeta$, $\sigma \to \omega(\sigma,\cdot)$ is real analytic. Furthermore $\omega(\sigma,\cdot)$ has the following properties:*

(i) *$\omega(\sigma,k)$ extends analytically in the upper half plane $\operatorname{Im} k > 0$, and it has no zeroes in $\operatorname{Im} k \geq 0$.*

(ii) $\overline{\omega(\sigma,k)} = \omega(\sigma,-k)\ \ \forall k \in \mathbb{R}$.

(iii) *For every $k \in \mathbb{R}$*

$$\omega(\sigma,k)\omega(\sigma,-k) = \frac{4(k^2+1)}{4k^2 + \sigma(k)\sigma(-k)} \ .$$

Proof. By Lemma 2.60, the Hilbert transform is a bounded linear operator from $H^M_{\zeta,\mathbb{C}}$ to $H^M_{\zeta,\mathbb{C}}$. By Proposition 2.28 it then follows that the map

$$\mathscr{S}^{M,N} \to H^M_\zeta \ , \quad \sigma \mapsto \mathcal{H}(l(\sigma,\cdot))$$

is real analytic as well. Since the exponential function is real analytic and $\partial_k\omega(\sigma,\cdot) = \frac{1}{2}\partial_k(l(\sigma,\cdot) + i\mathcal{H}(l(\sigma,\cdot)))\omega(\sigma,\cdot)$, one proves by induction that $\mathscr{S}^{M,N} \to W^M_\zeta$, $\sigma \to \omega(\sigma,\cdot)$ is real analytic. Properties (i)–(iii) are proved in [KT86, Section 4]. $\square$

Next we consider the map $\sigma \to \frac{1}{w(\sigma,\cdot)}$. The following proposition follows immediately from Proposition 2.29 and the definition $\frac{1}{w(\sigma,k)} = \frac{\omega(\sigma,k)}{2i(k+i)}$.

Proposition 2.30. *The map $\mathscr{S}^{M,N} \to H^{M-1}_{\mathbb{C}}$, $\sigma \to \frac{1}{w(\sigma,\cdot)}$ is real analytic. Furthermore the maps*

$$\mathscr{S}^{M,N} \to L^2 \ , \qquad \sigma \to \partial_k^n \frac{2ik}{w(\sigma,\cdot)} \ , \quad 1 \leq n \leq M$$

are real analytic. The function $\frac{1}{w(\sigma,\cdot)}$ fulfills

(i) $\overline{\left(\frac{1}{w(\sigma,k)}\right)} = \frac{1}{w(\sigma,-k)}$ *for every $k \in \mathbb{R}$.*

(ii) $\left|\frac{2ik}{w(\sigma,k)}\right| \leq 1$ *for every $k \in \mathbb{R}$.*

(iii) For every $k \in \mathbb{R}$

$$w(\sigma,k)w(\sigma,-k) = 4k^2 + \sigma(k)\sigma(-k) \ .$$

In particular $|w(\sigma,k)| > 0$ for every $k \in \mathbb{R}$ and $\sigma \in \mathscr{S}^{M,N}$.

Now we study the properties of $\rho_+(\sigma,\cdot)$ and $\rho_-(\sigma,\cdot)$ defined in formulas (2.69).

Proposition 2.31. *Let $N \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 4}$ be fixed. Then the maps $\mathscr{S}^{M,N} \to H^M_\zeta \cap L^2_N$, $\sigma \to \rho_\pm(\sigma,\cdot)$ are real analytic. There exists $C > 0$ so that $\|\rho_\pm(\sigma,\cdot)\|_{H^M_{\zeta,\mathbb{C}} \cap L^2_N} \leq C \|\sigma\|_{H^M_\zeta \cap L^2_N}$, where C depends locally uniformly on $\sigma \in \mathscr{S}^{M,N}$. Furthermore the following holds:*

(i) unitarity: $\tau(\sigma,k)\tau(\sigma,-k) + \rho_\pm(\sigma,k)\rho_\pm(\sigma,-k) = 1$ and $\rho_+(\sigma,k)\overline{\tau(\sigma,k)} + \overline{\rho_-(\sigma,k)}\tau(\sigma,k) = 0$ for every $k \in \mathbb{R}$.

(ii) reality: $\tau(\sigma,k) = \overline{\tau(\sigma,-k)}$, $\rho_\pm(\sigma,k) = \overline{\rho_\pm(\sigma,-k)}$;

(iii) analyticity: $\tau(\sigma,k)$ admits an analytic extension to $\{\mathrm{Im}\, k > 0\}$;

(iv) asymptotics: $\tau(\sigma,z) = 1 + O(1/|z|)$ as $|z| \to \infty$, $\mathrm{Im}\, z \geq 0$, and $\rho_\pm(\sigma,k) = O(1/k)$, as $|k| \to \infty$, k real;

(v) rate at $k = 0$: $|\tau(\sigma,z)| > 0$ for $z \neq 0$, $\mathrm{Im}\, z \geq 0$ and $|\rho_\pm(\sigma,k)| < 1$ for $k \neq 0$. Furthermore

$$\begin{aligned} \tau(\sigma,z) =& \alpha z + o(z), \quad \alpha \neq 0, \quad \mathrm{Im}\, z \geq 0 \\ 1 + \rho_\pm(\sigma,k) =& \beta_\pm k + o(k), \quad k \in \mathbb{R}; \end{aligned}$$

Proof. The real analyticity of the maps $\mathscr{S}^{M,N} \to H^M_\zeta \cap L^2_N$, $\sigma \to \rho_\pm(\sigma,\cdot)$ follows from Proposition 2.30 and the definition $\rho_\pm(\sigma,k) = \sigma(\mp,k)/w(\sigma,k)$ (see also the proof of Proposition 2.32). Since $\sigma \mapsto \frac{1}{w(\sigma,\cdot)}$ is real analytic, it is locally bounded, i.e., there exists $C > 0$ so that $\|\rho_\pm(\sigma,\cdot)\|_{H^M_{\zeta,\mathbb{C}} \cap L^2_N} \leq C \|\sigma\|_{H^M_\zeta \cap L^2_N}$, where C depends locally uniformly on $\sigma \in \mathscr{S}^{M,N}$. Properties $(i), (ii), (v)$ follow now by simple computations. Property $(iii) - (iv)$ are proved in [KT86, Lemma 4.1]. $\square$

Finally define the functions

$$R_\pm(\sigma,k) := 2ik\rho_\pm(\sigma,k) \ . \tag{2.74}$$

Proposition 2.32. *Let $N \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 4}$ be fixed. Then the maps $\mathscr{S}^{M,N} \to H^M_{\mathbb{C}} \cap L^2_N$, $\sigma \to R_\pm(\sigma,\cdot)$ are real analytic. There exists $C > 0$ so that $\|R_\pm(\sigma,\cdot)\|_{H^M_{\mathbb{C}} \cap L^2_N} \leq C \|\sigma\|_{H^M_\zeta \cap L^2_N}$, where C depends locally uniformly on $\sigma \in \mathscr{S}^{M,N}$. Furthermore the following holds:*

(i) $\overline{R_\pm(\sigma,k)} = R_\pm(\sigma,-k)$ *for every* $k \in \mathbb{R}$.

(ii) $|R_\pm(\sigma,k)| < 2|k|$ *for any* $k \in \mathbb{R}\setminus\{0\}$.

Proof. In order to prove the statements, we will use that $R_\pm(\sigma,k) = 2ik\frac{\sigma(\mp k)}{w(\sigma,k)}$. We will consider just R_-, since the analysis for R_+ is identical. To simplify the notation, we will denote $R_-(\sigma,\cdot) \equiv R(\sigma,\cdot)$.

By Proposition 2.30(ii), $|R(\sigma,k)| \leq |\sigma(k)|$, thus $R(\sigma,\cdot) \in L^2_N$. In order to prove that $R(\sigma,\cdot) \in H^M_{\mathbb{C}}$, take n derivatives $(1 \leq n \leq M)$ of $R(\sigma,\cdot)$ to get the identity

$$\partial_k^n R(\sigma,k) = \frac{2ik}{w(\sigma,k)}\partial_k^n\sigma(k) + \sum_{j=1}^{n-1}\binom{n}{j}\left(\partial_k^j\frac{2ik}{w(\sigma,k)}\right)\ \partial_k^{n-j}\sigma(k) + \left(\partial_k^n\frac{2ik}{w(\sigma,k)}\right)\sigma(k)\ . \qquad (2.75)$$

We show now that each term of the r.h.s. of the identity above is in L^2. Consider first the term $I_1 := \frac{2ik}{w(\sigma,k)}\partial_k^n\sigma(k)$. If $1 \leq n < M$, then $\partial_k^n\sigma \in L^2$ and $|2ik/w(\sigma,k)| \leq 1$, thus proving that $I_1 \in L^2$. If $n = M$, let χ be a smooth cut-off function with $\chi(k) \equiv 1$ in $[-1,1]$ and $\chi(k) \equiv 0$ in $\mathbb{R}\setminus[-2,2]$. Then one has

$$I_1 = \frac{1}{w(\sigma,k)}\chi(k)2ik\partial_k^M\sigma(k) + \frac{2ik}{w(\sigma,k)}(1-\chi(k))\partial_k^M\sigma(k)\ .$$

As $\sigma \in \mathscr{S}^{M,N}$ it follows that $k \mapsto \chi(k)2ik\partial_k^M\sigma(k)$ and $k \mapsto (1-\chi(k))\partial_k^M\sigma(k)$ are in L^2. By Proposition 2.30, $\frac{1}{w(\sigma,\cdot)}$ and $\frac{2ik}{w(\sigma,\cdot)}$ are in L^∞. Altogether it follows that $I_1 \in L^2$ for any $1 \leq n \leq M$.

Consider now $I_2 := \sum_{j=1}^{n-1}\binom{n}{j}\left(\partial_k^j\frac{2ik}{w(\sigma,k)}\right)\ \partial_k^{n-j}\sigma(k)$. By Proposition 2.30, $\left(\partial_k^j\frac{2ik}{w(\sigma,k)}\right) \in H^1_{\mathbb{C}}$ for every $1 \leq j \leq M-1$, thus by the Sobolev embedding theorem $\left(\partial_k^j\frac{2ik}{w(\sigma,k)}\right) \in L^\infty$ for every $1 \leq j \leq M-1$. As $\partial_k^{n-j}\sigma \in L^2$ for $1 \leq j \leq n-1 < M$, it follows that $I_2 \in L^2$ for any $1 \leq n \leq M$.

Finally consider $I_3 := \left(\partial_k^n\frac{2ik}{w(\sigma,k)}\right)\sigma(k)$. By Proposition 2.30, $\left(\partial_k^n\frac{2ik}{w(\sigma,k)}\right) \in L^2$ for any $1 \leq n \leq M$. Since $\sigma \in L^\infty$, $I_3 \in L^2$ for any $1 \leq n \leq M$.

Altogether we proved that $R(\sigma,\cdot) \in H^M_{\mathbb{C}} \cap L^2_N$. The claimed estimate on $\|R(\sigma,\cdot)\|_{H^M_{\mathbb{C}}\cap L^2_N}$ and item (i) and (ii) follow in a straightforward way. The real analyticity of the map $\mathscr{S}^{M,N} \to H^M_{\mathbb{C}}\cap L^2_N$, $\sigma \to R(\sigma,\cdot)$ follows by Proposition 2.30. $\square$

For $\sigma \in \mathscr{S}^{M,N}$, define the Fourier transforms

$$F_\pm(\sigma,y) := \mathcal{F}_\pm^{-1}(\rho_\pm(\sigma,\cdot))(y) = \frac{1}{\pi}\int_{\mathbb{R}}\rho_\pm(\sigma,k)e^{\pm 2iky}dk\ . \qquad (2.76)$$

Then

$$\pm\partial_y F_\pm(\sigma,y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} 2ik\rho_\pm(\sigma,k)e^{\pm 2iky}\ dk = \mathcal{F}_\pm^{-1}(R_\pm(\sigma,\cdot))(y)\ . \qquad (2.77)$$

In the next proposition we analyze the properties of the maps $\sigma \mapsto F_\pm(\sigma,\cdot)$.

Proposition 2.33. *Let* $N \in \mathbb{Z}_{\geq 0}$ *and* $M \in \mathbb{Z}_{\geq 4}$ *be fixed. Then the following holds true:*

(i) $\sigma \mapsto F_\pm(\sigma,\cdot)$ *are real analytic as maps from* $\mathscr{S}^{4,0}$ *to* $H^1 \cap L^2_3$*. Moreover there exists* $C > 0$ *so that* $\|F_\pm(\sigma,\cdot)\|_{H^1\cap L^2_3} \leq C\,\|\sigma\|_{H^M_\zeta}$*, where* C *depends locally uniformly on* $\sigma \in \mathscr{S}^{M,N}$.

(ii) $\sigma \mapsto F'_\pm(\sigma,\cdot)$ *are real analytic as maps from* $\mathscr{S}^{M,N}$ *to* $H^N \cap L^2_M$*. Moreover there exists* $C' > 0$ *so that* $\left\|F'_\pm(\sigma,\cdot)\right\|_{H^N\cap L^2_M} \leq C' \left\|\sigma\right\|_{H^M_\zeta \cap L^2_N}$*, where* C' *depends locally uniformly on* $\sigma \in \mathscr{S}^{M,N}$*.*

Proof. By Proposition 2.31, the map $\mathscr{S}^{4,0} \to H^3_{\mathbb{C}} \cap L^2_1$, $\sigma \to \rho_\pm(\sigma,\cdot)$ is real analytic. Thus item (i) follows by the properties of the Fourier transform. By Proposition 2.31 (ii), $F_\pm(\sigma,\cdot) = \mathcal{F}^{-1}_\pm(\rho_\pm)$ is real valued. Item (ii) follows from (2.77) and the characterizations

$$R_\pm \in H^M_{\mathbb{C}} \iff \mathcal{F}^{-1}_\pm(R_\pm) \in L^2_M \qquad \text{and} \qquad R_\pm \in L^2_N \iff \mathcal{F}^{-1}_\pm(R_\pm) \in H^N_{\mathbb{C}} \ . \tag{2.78}$$

The claimed estimates follow from the properties of the Fourier transform, Proposition 2.31 and Proposition 2.32. □

We are finally able to prove that there exists a potential $q \in \mathcal{Q}$ with prescribed scattering coefficient $\sigma \in \mathscr{S}^{M,N}$. More precisely the following theorem holds.

Theorem 2.34. *Let* $N \in \mathbb{Z}_{\geq 0}$*,* $M \in \mathbb{Z}_{\geq 4}$ *and* $\sigma \in \mathscr{S}^{M,N}$ *be fixed. Then there exists a potential* $q \in \mathcal{Q}$ *such that* $S(q,\cdot) = \sigma$*.*

Proof. Let $\rho_\pm := \rho_\pm(\sigma,\cdot)$ and $\tau := \tau(\sigma,\cdot)$ be given by formula (2.69). Let $F_\pm(\sigma,\cdot)$ be defined as in (2.76). By Proposition 2.33 it follows that $F_\pm(\sigma,\cdot)$ are absolutely continuous and $F'_\pm(\sigma,\cdot) \in H^N \cap L^2_M$. As $M \geq 4$ it follows that

$$\int_{-\infty}^{\infty} (1+x^2)|F'_\pm(\sigma,x)|\, dx < \infty \ . \tag{2.79}$$

The main theorem in inverse scattering [Fad64] assures that if (2.79) and item (i)–(v) of Proposition 2.32 hold, then there exists a potential $q \in \mathcal{Q}$ such that $r_\pm(q,\cdot) = \rho_\pm$ and $t(q,\cdot) = \tau$, where $r_\pm$ and t are the reflection respectively transmission coefficients defined in (2.5). From the formulas (2.69) it follows that $S(q,\cdot) = \sigma$. □

It remains to show that $q \in \mathcal{Q}^{N,M}$ and that the map $S^{-1} : \mathscr{S}^{M,N} \to \mathcal{Q}^{N,M}$ is real analytic. We take here a different approach then [KT86]. In [KT86] the authors show that the map S is complex differentiable and its differential d_qS is bounded invertible. Here instead we reconstruct q by solving the Gelfand-Levitan-Marchenko equations and we show that the inverse map $\mathscr{S}^{M,N} \to \mathcal{Q}^{N,M}$, $\sigma \mapsto q$ is real analytic. We outline briefly the procedure. Given two reflection coefficients $\rho_\pm$ satisfying items (i)–(v) of Proposition 2.31 and arbitrary real numbers $c_+ \leq c_-$, it is possible to construct a potential q_+ on $[c_+,\infty)$ using ρ_+ and a potential q_- on $(-\infty, c_-]$ using ρ_-, such that q_+ and q_- coincide on the intersection of their domains, i.e., $q_+|_{[c_+,c_-]} = q_-|_{[c_+,c_-]}$. Hence q defined on $\mathbb{R}$ by $q|_{[c_+,+\infty)} = q_+$ and $q|_{(-\infty,c_-]} = q_-$ is well defined, $q \in \mathcal{Q}$ and $r_\pm(q,\cdot) = \rho_\pm$, i.e., ρ_+ and ρ_- are the reflection coefficients of the potential q [Fad64, Mar86, DT79]. We postpone the details of this procedure to the next section.

4.1 Gelfand-Levitan-Marchenko equation

In this section we prove how to construct for any $\sigma \in \mathscr{S}^{M,N}$ two potentials q_+ and q_- with $q_+ \in H^N_{x\geq c} \cap L^2_{M,x\geq c}$ respectively $q_- \in H^N_{x\leq c} \cap L^2_{M,x\leq c}$, where for any $c \in \mathbb{R}$ and $1 \leq p \leq \infty$

$$L^p_{x\geq c} := \left\{ f : [c,+\infty) \to \mathbb{C} : \ \|f\|_{L^p_{x\geq c}} < \infty \right\} \ , \tag{2.80}$$

where $\|f\|_{L^p_{x\geq c}} := \left(\int_c^{+\infty} |f(x)|^p \, dx\right)^{1/p}$ for $1 \leq p < \infty$ and $\|f\|_{L^\infty_{x\geq c}} := \operatorname{ess\,sup}_{x\geq c} |f(x)|$. For any integer $N \geq 1$ define

$$H^N_{x\geq c} := \left\{ f : [c, +\infty) \to \mathbb{R} : \|f\|_{H^N_{x\geq c}} < \infty \right\}, \qquad \|f\|^2_{H^N_{x\geq c}} := \sum_{j=0}^{N} \|\partial_x^n f\|^2_{L^2_{x\geq c}} , \tag{2.81}$$

and for any real number $M \geq 1$ define

$$L^2_{M,x\geq c} := \left\{ f : [c, +\infty) \to \mathbb{C} : \|f\|_{L^2_{M,x\geq c}} < \infty \right\}, \qquad \|f\|_{L^2_{M,x\geq c}} = \left\| \langle x \rangle^M f \right\|_{L^2_{x\geq c}} , \tag{2.82}$$

where $\langle x \rangle := (1 + x^2)^{1/2}$. We will write $H^N_{\mathbb{C},x\geq c}$ for the complexification of $H^N_{x\geq c}$. For $1 \leq \alpha, \beta \leq \infty$, we define

$$L^\alpha_{x\geq c} L^\beta_{y\geq 0} := \left\{ f : [c, +\infty) \times [0, +\infty) \to \mathbb{C} : \|f\|_{L^\alpha_{x\geq c} L^\beta_{y\geq 0}} < \infty \right\} \ ,$$

where $\|f\|_{L^\alpha_{x\geq c} L^\beta_{y\geq 0}} := \left(\int_c^{+\infty} \|f(x, \cdot)\|^\alpha_{L^\beta_{y\geq 0}} \, dx \right)^{1/\alpha}$. Analogously one defines the spaces $L^p_{x\leq c}$, $H^N_{x\leq c}$, $L^2_{M,x\leq c}$ and $L^\alpha_{x\leq c} L^\beta_{y\leq 0}$, *mutatis mutandis.*
Let us denote by $C^0_{y\geq 0} := C^0([0, \infty), \mathbb{C})$ and by $C^0_{x\geq c, y\geq 0} := C^0([c, \infty) \times [0, \infty), \mathbb{C})$. Finally we denote by $C^0_{x\geq c} L^2_{y\geq 0} := C^0([c, \infty), L^2_{y\geq 0})$ the set of continuous functions on $[c, \infty)$ taking value in $L^2_{y\geq 0}$.

The potentials q_+ and q_- mentioned at the beginning of this section are constructed by solving an integral equation, known in literature as the *Gelfand-Levitan-Marchenko equation*, which we are now going to described in more detail.

Given $\sigma \in \mathscr{S}$, define the functions $F_\pm(\sigma, \cdot)$ as in (2.76). See Proposition 2.33 for the analytical properties of the maps $\sigma \to F_\pm(\sigma, \cdot)$. To have a more compact notation, in the following we will denote $F_{\pm,\sigma} := F_\pm(\sigma, \cdot)$.

Remark 2.35. *From the decay properties of $F'_{\pm,\sigma}$ one deduces corresponding decay properties of $F_{\pm,\sigma}$. Indeed one has*

$$\langle x \rangle^m F'_\pm \in L^2_{x\geq c} \Rightarrow \langle x \rangle^{m-1} F'_\pm \in L^1_{x\geq c} \Rightarrow x^{m-2} F_\pm \in L^1_{x\geq c} \ , \quad \forall \, m \geq 2 \ . \tag{2.83}$$

The Gelfand-Levitan-Marchenko equations are the integral equations given by

$$F_{+,\sigma}(x + y) + E_{+,\sigma}(x, y) + \int_0^{+\infty} F_{+,\sigma}(x + y + z) E_{+,\sigma}(x, z) dz = 0, \qquad y \geq 0 \tag{2.84}$$

$$F_{-,\sigma}(x + y) + E_{-,\sigma}(x, y) + \int_{-\infty}^{0} F_{-,\sigma}(x + y + z) E_{-,\sigma}(x, z) dz = 0, \qquad y \leq 0 \tag{2.85}$$

where $E_{\pm,\sigma}(x, y)$ are the unknown functions and $F_{\pm,\sigma}$ are given and uniquely determined by σ through formula (2.76). If (2.84) and (2.85) have solutions with enough regularity, then one defines the potentials q_+ and q_- through the well-known formula – [Fad64]

$$q_+(x) = -\partial_x E_{+,\sigma}(x, 0), \quad \forall \, c_+ \leq x < \infty \ , \qquad q_-(x) = \partial_x E_{-,\sigma}(x, 0), \quad \forall -\infty < x \leq c_- \ . \tag{2.86}$$

The main purpose of this section is to study the maps $\mathcal{R}_{\pm,c}$ defined by

$$\sigma \mapsto \mathcal{R}_{\pm,c}(\sigma), \qquad \mathcal{R}_{\pm,c}(\sigma)(x) := \mp\partial_x E_{\pm,\sigma}(x,0), \qquad x \in [c, \pm\infty) \ . \tag{2.87}$$

Theorem 2.36. *Fix $N \in \mathbb{Z}_{\geq 0}$, $M \in \mathbb{Z}_{\geq 4}$ and $c \in \mathbb{R}$. Then the maps $\mathcal{R}_{+,c}$ $[\mathcal{R}_{-,c}]$ are well defined on $\mathscr{S}^{M,N}$ and take values in $H^N_{x\geq c} \cap L^2_{M,x\geq c}$ $[H^N_{x\leq c} \cap L^2_{M,x\leq c}]$. As such they are real analytic.*

In order to prove Theorem 2.36 we look for solutions of (2.84) and (2.85) of the form

$$E_{\pm,\sigma}(x,y) \equiv -F_{\pm,\sigma}(x+y) + B_{\pm,\sigma}(x,y) \tag{2.88}$$

where $B_{\pm,\sigma}(x,y)$ are to be determined. Inserting the ansatz (2.88) into the Gelfand-Levitan-Marchenko equations (2.84), (2.85), one gets

$$B_{+,\sigma}(x,y) + \int_0^{+\infty} F_{+,\sigma}(x+y+z)B_{+,\sigma}(x,z)dz = \int_0^{+\infty} F_{+,\sigma}(x+y+z)F_{+,\sigma}(x+z)\,dz, \qquad y \geq 0 \ , \tag{2.89}$$

$$B_{-,\sigma}(x,y) + \int_{-\infty}^{0} F_{-,\sigma}(x+y+z)B_{-,\sigma}(x,z)dz = \int_{-\infty}^{0} F_{-,\sigma}(x+y+z)F_{-,\sigma}(x+z)\,dz, \qquad y \leq 0. \tag{2.90}$$

We will prove in Lemma 2.38 below that there exists a solution $B_{+,\sigma}$ of (2.89) and a solution $B_{-,\sigma}$ of (2.90) with $\partial_x B_{+,\sigma}(\cdot,0) \in H^1_{x\geq c}$ respectively $\partial_x B_{-,\sigma}(\cdot,0) \in H^1_{x\leq c}$. By (2.86) we get therefore

$$q_+ = \partial_x F_{+,\sigma} - \partial_x B_{+,\sigma}(\cdot,0) \quad \forall\, c \leq x < \infty, \qquad q_- = -\partial_x F_{-,\sigma} + \partial_x B_{-,\sigma}(\cdot,0) \quad \forall -\infty < x \leq c \ . \tag{2.91}$$

Define the maps

$$\mathcal{B}_{\pm,c} : \sigma \mapsto \mathcal{B}_{\pm,c}(\sigma)$$

as

$$\mathcal{B}_{+,c}(\sigma)(x) := -\partial_x B_{+,\sigma}(x,0) \quad \forall\, x \geq c \qquad \text{and} \qquad \mathcal{B}_{-,c}(\sigma)(x) := \partial_x B_{-,\sigma}(x,0) \quad \forall\, x \leq c \ , \tag{2.92}$$

with $B_{\pm,\sigma}(x,y) := E_{\pm,\sigma}(x,y) + F_{\pm,\sigma}(x,y)$ as in (2.88). Now we study analytic properties of the maps $\mathcal{B}_{\pm,c}$ in case the scattering coefficient σ belongs to $\mathscr{S}^{4,N}$ with arbitrary $N \in \mathbb{Z}_{\geq 0}$. Later we will treat the case where $\sigma \in \mathscr{S}^{M,0}$, $M \in \mathbb{Z}_{\geq 4}$.

Proposition 2.37. *Fix $N \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{R}$. Then $\mathcal{B}_{+,c}$ $[\mathcal{B}_{-,c}]$ is real analytic as a map from $\mathscr{S}^{4,N}$ to $H^N_{x\geq c}$ $[H^N_{x\leq c}]$. Moreover*

$$\|\mathcal{B}_{+,c}(\sigma)\|_{H^N_{x\geq c}} \ , \ \|\mathcal{B}_{-,c}(\sigma)\|_{H^N_{x\leq c}} \ \leq K \,\|\sigma\|^2_{H^4_{\zeta,\mathbb{C}}\cap L^2_N}$$

where $K > 0$ is a constant which can be chosen locally uniformly in $\sigma \in \mathscr{S}^{4,N}$.

The main ingredient of the proof of Proposition 2.37 is a detailed analysis of the solutions of the integral equations (2.89)-(2.90), which we rewrite as

$$\left(Id + \mathcal{K}^{\pm}_{x,\sigma}\right)[B_{\pm,\sigma}(x,\cdot)](y) = f_{\pm,\sigma}(x,y) \tag{2.93}$$

where for every $x \in \mathbb{R}$ fixed, the two operators $\mathcal{K}^+_{x,\sigma} : L^2_{y\geq 0} \to L^2_{y\geq 0}$ and $\mathcal{K}^-_{x,\sigma} : L^2_{y\leq 0} \to L^2_{y\leq 0}$ are defined by

$$\mathcal{K}^+_{x,\sigma}[f](y) := \int_0^{+\infty} F_{+,\sigma}(x+y+z) f(z)\, dz \ , \qquad f \in L^2_{y\geq 0} \ , \tag{2.94}$$

$$\mathcal{K}^-_{x,\sigma}[f](y) := \int_{-\infty}^{0} F_{-,\sigma}(x+y+z) f(z)\, dz \ , \qquad f \in L^2_{y\leq 0} \ , \tag{2.95}$$

and the functions $f_{\pm,\sigma}$ are defined by

$$f_{\pm,\sigma}(x,y) := \pm \int_0^{\pm\infty} F_{\pm,\sigma}(x+y+z) F_{\pm,\sigma}(x+z)\, dz \ . \tag{2.96}$$

As the claimed statements for $\mathcal{B}_{+,c}$ and $\mathcal{B}_{-,c}$ can be proved in a similar way we consider $\mathcal{B}_{+,c}$ only. To simplify notation, in the following we omit the subscript " + ". In particular we write $B_\sigma \equiv B_{+,\sigma}$, $F_\sigma \equiv F_{+,\sigma}$, $f_\sigma \equiv f_{+,\sigma}$ and $\mathcal{K}_{x,\sigma} \equiv \mathcal{K}^+_{x,\sigma}$.

We give the following definition: a function $h_\sigma : [c,\infty) \times [0,\infty) \to \mathbb{R}$, which depends on $\sigma \in \mathscr{S}^{4,N}$, will be said to satisfy (P) if the following holds true:

$(P1)$ $h_\sigma \in C^0_{x\geq c} L^2_{y\geq 0} \cap L^2_{x\geq c} L^2_{y\geq 0} \cap C^0_{x\geq c, y\geq 0}$. Finally $h_\sigma(\cdot, 0) \in L^2_{x\geq c}$.

$(P2)$ There exists a constant $K_c > 0$, which depends locally uniformly on $\sigma \in H^4_{\zeta,\mathbb{C}} \cap L^2_N$, such that

$$\|h_\sigma\|_{L^2_{x\geq c} L^2_{y\geq 0}} + \|h_\sigma(\cdot, 0)\|_{L^2_{x\geq c}} \leq K_c \|\sigma\|^2_{H^4_{\zeta,\mathbb{C}} \cap L^2_N} \ . \tag{2.97}$$

$(P3)$ $\sigma \mapsto h_\sigma$ $[\sigma \mapsto h_\sigma(\cdot,0)]$ is real analytic as a map from $\mathscr{S}^{4,N}$ to $L^2_{x\geq c} L^2_{y\geq 0}$ $[L^2_{x\geq c}]$.

We have the following lemma:

Lemma 2.38. *Fix $N \geq 0$ and $c \in \mathbb{R}$. For every $\sigma \in \mathscr{S}^{4,N}$ equation (2.89) has a unique solution $B_\sigma \in C^0_{x\geq c} L^2_{y\geq 0} \cap L^2_{x\geq c} L^2_{y\geq 0}$. Moreover for all integers $n_1, n_2 \geq 0$ with $n_1 + n_2 \leq N+1$, the function $\partial_x^{n_1} \partial_y^{n_2} B_\sigma$ satisfies (P).*

Proof. Let $N \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{R}$ be fixed. The proof is by induction on $j_1 + j_2 = n$, $0 \leq n \leq N$. For each n we prove that $\partial_x^{j_1} \partial_y^{j_2} B_\sigma$ and its derivatives $\partial_x^{j_1+1} \partial_y^{j_2} B_\sigma$, $\partial_x^{j_1} \partial_y^{j_2+1} B_\sigma$ satisfy (P). Thus the claim follows.

Case $n = 0$. Then $j_1 = j_2 = 0$. We need to prove existence and uniqueness of the solution of equation (2.93). By Lemma 2.55 [Proposition 2.33 and Lemma 2.54] the function f_σ and its derivatives $\partial_x f_\sigma$, $\partial_y f_\sigma$ $[F_\sigma]$ satisfy assumption (P) [(H)– cf Appendix C]. Thus by Lemma 2.53 (i) it follows that $B_\sigma = (Id + \mathcal{K}_\sigma)^{-1} f_\sigma$ and its derivatives $\partial_x B_\sigma$, $\partial_y B_\sigma$ satisfy (P).

Note that if $N = 0$ the lemma is proved. Thus in the following we assume $N \geq 1$.

Case $n - 1 \rightsquigarrow n$. Let $j_1 + j_2 = n$. By the induction assumption we already know that $\partial_x^{j_1} \partial_y^{j_2} B_\sigma$ satisfies (P). By Lemma 2.53 it follows that $\partial_x^{j_1} \partial_y^{j_2} B_\sigma$ satisfies

$$\begin{cases} (Id + \mathcal{K}_{x,\sigma})[\partial_x^n B_\sigma(x,\cdot)](y) = f_\sigma^{n,0}(x,y) & \text{if } j_2 = 0 \ , \\ \partial_x^{j_1} \partial_y^{j_2} B_\sigma(x,y) = f_\sigma^{j_1,j_2}(x,y) & \text{if } j_2 > 0 \ , \end{cases} \tag{2.98}$$

where

$$
\begin{aligned}
f_\sigma^{n,0}(x,y) &:= \partial_x^n f_\sigma(x,y) - \sum_{l=1}^{n} \binom{n}{l} \int_0^{+\infty} \partial_x^l F_\sigma(x+y+z)\, \partial_x^{n-l} B_\sigma(x,z)\, dz \ , \\
f_\sigma^{j_1,j_2}(x,y) &:= \partial_x^{j_1} \partial_y^{j_2} f_\sigma(x,y) - \sum_{l=0}^{j_1} \binom{j_1}{l} \int_0^{+\infty} \partial_z^{j_2+l} F_\sigma(x+y+z)\, \partial_x^{j_1-l} B_\sigma(x,z)\, dz \ .
\end{aligned}
\tag{2.99}
$$

In order to prove the induction step, we show in Lemma 2.57 that for any $j_1 + j_2 = n$, $0 \leq n \leq N$, $f_\sigma^{j_1,j_2}$ and its derivatives $\partial_y f_\sigma^{j_1,j_2}$, $\partial_x f_\sigma^{j_1,j_2}$ satisfy (P). In view of identities (2.98) and Lemma 2.53 (i), it follows that $\partial_x^{j_1}\partial_y^{j_2} B_\sigma$ and its derivatives $\partial_x^{j_1+1}\partial_y^{j_2} B_\sigma$ and $\partial_x^{j_1}\partial_y^{j_2+1} B_\sigma$ satisfy (P), thus proving the induction step. $\square$

Lemma 2.38 implies in a straightforward way Proposition 2.37.

Proof of Proposition 2.37. By Lemma 2.38, $\partial_x^n B_\sigma$ satisfies (P) for every $1 \leq n \leq N+1$. In particular for every $1 \leq n \leq N+1$, $\sigma \mapsto \partial_x^n B_\sigma(\cdot, 0)$ is real analytic as a map from $\mathscr{S}^{4,N}$ to $L^2_{x\geq c}$ and $\|\partial_x^n B_\sigma(\cdot,0)\|_{L^2_{x\geq c}} \leq K_c \|\sigma\|^2_{H^4_{\zeta,\mathbb{C}} \cap L^2_N}$. Thus the map $\sigma \to -\partial_x B_\sigma(\cdot,0)$ is real analytic as a map from $\mathscr{S}^{4,N}$ to $H^N_{x\geq c}$. The claimed estimate follows in a straightforward way. $\square$

In the next result we study the case $\sigma \in \mathscr{S}^{M,0}$ for arbitrary $M \geq 4$.

Proposition 2.39. *Fix $M \in \mathbb{Z}_{\geq 4}$ and $c \in \mathbb{R}$. For any $\sigma \in \mathscr{S}^{M,0}$ the equations* (2.84) *and* (2.85) *admit solutions $E_{\pm,\sigma}$. The maps $\mathcal{R}_{+,c}$ $[\mathcal{R}_{-,c}]$, defined by* (2.87), *are real analytic as maps from $\mathscr{S}^{M,0}$ to $L^2_{M,x\geq c}$ $[L^2_{M,x\leq c}]$. Moreover $\|\mathcal{R}_{+,c}(\sigma)\|_{L^2_{M,x\geq c}}$, $\|\mathcal{R}_{-,c}(\sigma)\|_{L^2_{M,x\leq c}} \leq K_c \|\sigma\|_{H^M_{\zeta,\mathbb{C}}}$, where $K_c > 0$ can be chosen locally uniformly in $\sigma \in \mathscr{S}^{M,0}$.*

Proof. We prove the result just for $\mathcal{R}_{+,c}$, since for $\mathcal{R}_{-,c}$ the proof is analogous. As before, we suppress the subscript "+" from the various objects.
Consider the Gelfand-Levitan-Marchenko equation (2.84). Multiply it by $\langle x\rangle^{M-3/2}$ to obtain

$$
(Id + \mathcal{K}_{x,\sigma}) \left[\langle x\rangle^{M-3/2} E_\sigma(x,y) \right] = -\langle x\rangle^{M-3/2} F_\sigma(x+y). \tag{2.100}
$$

The function

$$
h_\sigma(x,y) := -\langle x\rangle^{M-3/2} F_\sigma(x+y) \ ,
$$

satisfies $h_\sigma(x,\cdot) \in L^2_{y\geq 0}$ and one checks that $h_\sigma \in C^0_{x\geq c} L^2_{y\geq 0} \cap C^0_{x\geq c, y\geq 0}$. We show now that $h_\sigma \in L^2_{x\geq c} L^2_{y\geq 0}$. By Lemma 2.42 $(A3)$ and Proposition 2.33 for $N=0$ it follows that

$$
\left\| \langle x\rangle^{M-3/2} h_\sigma \right\|^2_{L^2_{x\geq c} L^2_{y\geq 0}} \leq K_c \int_c^{+\infty} \langle x\rangle^{2M-2} |F_\sigma(x)|^2\, dx \leq K_c \left\| \langle x\rangle^M F'_\sigma \right\|^2_{L^2_{x\geq c}} \leq K_c \|\sigma\|^2_{H^M_{\zeta,\mathbb{C}}} \ .
$$

Consider now $h_\sigma(x,0) = -\langle x\rangle^{M-3/2} F_\sigma(x)$. By (2.83) it follows that $h_\sigma(\cdot,0) \in L^2_{x\geq c}$. Finally the map $\sigma \mapsto h_\sigma$ $[\sigma \mapsto h_\sigma(\cdot,0)]$ is real analytic as a map from $\mathscr{S}^{M,0}$ to $L^2_{x\geq c} L^2_{y\geq 0}$ $[L^2_{M-3/2,x\geq c}]$.

Proceeding as in the proof of Lemma 2.51, one shows that there exists a solution E_σ of equation (2.84) which satisfies (i) $\langle x\rangle^{M-3/2}E_\sigma \in C^0_{x\geq c}L^2_{y\geq 0}\cap L^2_{x\geq c}L^2_{y\geq 0}$, $\langle x\rangle^{M-3/2}E_\sigma(x,\cdot)\in C^0_{y\geq 0}$, $\langle\cdot\rangle^{M-3/2}E_\sigma(\cdot,0)\in L^2_{x\geq c}$, (ii) $\left\|\langle x\rangle^{M-3/2}E_\sigma\right\|_{L^2_{x\geq c}L^2_{y\geq 0}}\leq K_c\left\|\sigma\right\|_{H^M_{\zeta,\mathbb{C}}}$, (iii) $\sigma\mapsto\langle x\rangle^{M-3/2}E_\sigma$ $[\sigma\mapsto E_\sigma(\cdot,0)]$ is real analytic as a map from $\mathscr{S}^{M,0}$ to $L^2_{x\geq c}L^2_{y\geq 0}$ $[L^2_{M-3/2,x\geq c}]$. Furthermore its derivative $\partial_x E_\sigma$ satisfies the integral equation

$$(Id+\mathcal{K}_{x,\sigma})\left(\partial_x E_\sigma(x,y)\right)=-F'_\sigma(x+y)-\int_0^{+\infty}F'_\sigma(x+y+z)\;E_\sigma(x,z)\,dz. \tag{2.101}$$

Multiply the equation above by $\langle x\rangle^{M-3/2}$, to obtain $(Id+\mathcal{K}_\sigma)\left(\langle x\rangle^{M-3/2}\partial_x E_\sigma\right)=\tilde{h}_\sigma$, where

$$\tilde{h}_\sigma(x,y):=-\langle x\rangle^{M-3/2}h'_\sigma(x,y)-\int_0^{+\infty}F'_\sigma(x+y+z)\,\langle x\rangle^{M-3/2}E_\sigma(x,z)\,dz\;. \tag{2.102}$$

where $h'_\sigma(x,y):=F'_\sigma(x+y)$. We claim that $\tilde{h}_\sigma\in L^2_{x\geq c}L^2_{y\geq 0}$ and $\sigma\mapsto\tilde{h}_\sigma$ is real analytic as a map $\mathscr{S}^{M,0}\to L^2_{x\geq c}L^2_{y\geq 0}$. By Lemma 2.42 $(A0)$ the first term of (2.102) satisfies

$$\left\|\langle x\rangle^{M-3/2}h'_\sigma\right\|_{L^2_{x\geq c}L^2_{y\geq 0}}\leq K_c\left\|\langle x\rangle^{M-1}F'_\sigma\right\|_{L^2_{x\geq c}}\leq K_c\left\|\sigma\right\|_{H^M_{\zeta,\mathbb{C}}}\;,$$

and by Lemma 2.42 $(A1)$ the second term of (2.102) satisfies

$$\left\|\int_0^{+\infty}F'_\sigma(x+y+z)\,\langle x\rangle^{M-3/2}E_\sigma(x,z)\,dz\right\|_{L^2_{x\geq c}L^2_{y\geq 0}}\leq\left\|F'_\sigma\right\|_{L^1}\left\|\langle x\rangle^{M-3/2}E_\sigma\right\|_{L^2_{x\geq c}L^2_{y\geq 0}}\leq K_c\left\|\sigma\right\|^2_{H^M_{\zeta,\mathbb{C}}}\;.$$

Moreover $\sigma\mapsto\tilde{h}_\sigma$ is real analytic as a map from $\mathscr{S}^{M,0}$ to $L^2_{x\geq c}L^2_{y\geq 0}$, being composition of real analytic maps.

Thus, by Lemma 2.51, it follows that $\langle x\rangle^{M-3/2}\partial_x E_\sigma\in L^2_{x\geq c}L^2_{y\geq 0}$, $\left\|\langle x\rangle^{M-3/2}\partial_x E_\sigma\right\|_{L^2_{x\geq c}L^2_{y\geq 0}}\leq K_c\left\|\sigma\right\|_{H^M_{\zeta,\mathbb{C}}}$ and $\sigma\mapsto\langle\cdot\rangle^{M-3/2}\partial_x E_\sigma$ is real analytic as a map from $\mathscr{S}^{M,0}$ to $L^2_{x\geq c}L^2_{y\geq 0}$.
Consider now equation (2.84). Evaluate it at $y=0$ to get

$$E_\sigma(x,0)=-F_\sigma(x)-\int_0^{+\infty}F_\sigma(x+z)E_\sigma(x,z)\,dz\;.$$

Take the x-derivative of the equation above and multiply it by $\langle x\rangle^M$ to obtain

$$\begin{aligned}\langle x\rangle^M\partial_x E_\sigma(x,0)=&-\langle x\rangle^M F'_\sigma(x)-\int_0^{+\infty}\langle x\rangle^{3/2}F'_\sigma(x+z)\langle x\rangle^{M-3/2}E_\sigma(x,z)\,dz\\&-\int_0^{+\infty}\langle x\rangle^{3/2}F_\sigma(x+z)\langle x\rangle^{M-3/2}\partial_x E_\sigma(x,z)\,dz\;.\end{aligned}$$

We prove now that $\partial_x E_\sigma(\cdot, 0) \in L^2_{M,x\geq c}$ and $\sigma \mapsto \partial_x E_\sigma(\cdot, 0)$ is real analytic as a map from $\mathscr{S}^{M,0}$ to $L^2_{M,x\geq c}$. The result follows by Proposition 2.33 and Lemma 2.42 ($A2$). Indeed one has that $\sigma \mapsto F'_\sigma$ $[\sigma \mapsto F_\sigma]$ is real analytic as a map from $\mathscr{S}^{M,0}$ to L^2_M $[L^2_{3/2}]$, and we proved above that $\sigma \mapsto \langle\cdot\rangle^{M-3/2} E_\sigma$ and $\sigma \mapsto \langle\cdot\rangle^{M-3/2}\partial_x E_\sigma$ are real analytic as maps from $\mathscr{S}^{M,0}$ to $L^2_{x\geq c} L^2_{y\geq 0}$. ☐

Combining the results of Proposition 2.37 and Proposition 2.39, we can prove Theorem 2.36.

Proof of Theorem 2.36. It follows from Proposition 2.33, Proposition 2.37 and Proposition 2.39 by restricting the scattering maps $\mathcal{R}_{\pm,c}$ to the spaces $\mathscr{S}^{M,N} = \mathscr{S}^{4,N} \cap \mathscr{S}^{M,0}$. ☐

Using the results of Theorem 2.36 and Theorem 2.34 we can prove Theorem 2.27, showing that $S^{-1} : \mathscr{S}^{N,M} \to \mathcal{Q}^{N,M}$ is real analytic.

Proof of Theorem 2.27. Let $\sigma \in \mathscr{S}^{M,N}$. By Theorem 2.34 there exists $q \in \mathcal{Q}$ with $S(q, \cdot) = \sigma$. Now let $c_+ \leq c \leq c_-$ be arbitrary real numbers and consider $\mathcal{R}_{+,c_+}(\sigma)$ and $\mathcal{R}_{-,c_-}(\sigma)$, where $\mathcal{R}_{\pm,c_\pm}$ are defined in (2.87). By classical inverse scattering theory [Fad64], [Mar86] the following holds:

(i) $\mathcal{R}_{+,c_+}(\sigma)\big|_{x\in[c_+,c]} = \mathcal{R}_{-,c_-}(\sigma)\big|_{x\in[c,c_-]}$,

(ii) the potential q_c defined by

$$q_c := \mathcal{R}_{+,c_+}(\sigma)\mathbb{1}_{[c,\infty)} + \mathcal{R}_{-,c_-}(\sigma)\mathbb{1}_{(-\infty,c]} \tag{2.103}$$

is in $\mathcal{Q}$ and satisfies $r_+(q_c, \cdot) = \rho_+(\sigma, \cdot)$, $r_-(q_c, \cdot) = \rho_-(\sigma, \cdot)$ and $t(q_c, \cdot) = \tau(\sigma, \cdot)$. Thus by formulas (2.5) and (2.69) it follows that $S(q_c, \cdot) = \sigma$.

Since S is 1-1 it follows that $q_c \equiv q$. Finally, by Theorem 2.36, $\mathscr{S}^{M,N} \to H^N_{x\geq c_+} \cap L^2_{M,x\geq c_+}$, $\sigma \mapsto \mathcal{R}_{+,c_+}(\sigma)$ and $\mathscr{S}^{M,N} \to H^N_{x\leq c_-} \cap L^2_{M,x\leq c_-}$, $\sigma \mapsto \mathcal{R}_{-,c_-}(\sigma)$ are real analytic. It follows that $q \in H^N \cap L^2_M$ and the map $S^{-1} : \sigma \to q$ is real analytic. ☐

5 Proof of Corollary 2.2 and Theorem 2.3

This section is devoted to the proof of Corollary 2.2 and Theorem 2.3. Both results are easy applications of Theorem 2.1.

Proof of Corollary 2.2. Let $N \geq 0$, $M \geq 4$ be fixed integers. Fix $q \in \mathcal{Q}^{N,M}$. By Theorem 2.1 the scattering map $S(q, \cdot)$ is in $\mathscr{S}^{M,N}$. Furthermore by the definition (2.10) of $I(q, k)$ there exists a constant $C > 0$ such that for any $|k| \geq 1$

$$|I(q, k)| \leq \frac{C|S(q, k)|^2}{|k|} \ .$$

In particular $I(q, \cdot) \in L^1_{2N+1}([1, \infty), \mathbb{R})$. By the real analyticity of the map $q \mapsto S(q, \cdot)$, it follows that $\mathcal{Q}^{N,M} \to L^1_{2N+1}([1, \infty), \mathbb{R})$, $q \mapsto I(q, \cdot)|_{[1,\infty)}$ is real analytic.

Now let us analyze $I(q, k)$ for $0 \leq k \leq 1$. By the definition (2.10) of $I(q, k)$ one has

$$I(q, k) + \frac{k}{\pi} \log\left(\frac{4k^2}{4(k^2 + 1)}\right) = -\frac{k}{\pi} \log\left(\frac{4(k^2 + 1)}{4k^2 + S(q, k)S(q, -k)}\right) \ .$$

By Proposition 2.28, the map $\mathscr{S}^{M,N} \to H^M_\zeta([0,1],\mathbb{R})$, $\sigma \to l(\sigma,k) := \log\left(\frac{4(k^2+1)}{4k^2+\sigma(k)\sigma(-k)}\right)$ is real analytic.
Thus also the map $\mathcal{Q}^{N,M} \to H^M_\zeta([0,1],\mathbb{R})$, $q \to l(S(q),\cdot)$ is real analytic, being composition of real analytic maps. It follows that the map $q \mapsto -\frac{k}{\pi} l(S(q),k)$ is real analytic as a map from $\mathcal{Q}^{N,M}$ to $H^M([0,1],\mathbb{R})$. $\square$

For $t \in \mathbb{R}$ and $\sigma \in H^1_{\mathbb{C}}$, let us denote by

$$\Omega^t(\sigma)(k) := e^{-i8k^3 t}\sigma(k) \ . \tag{2.104}$$

We prove the following lemma.

Lemma 2.40. *Let N, M be integers with $N \geq 2M \geq 2$. Let $\sigma \in \mathscr{S}^{M,N}$. Then $\Omega^t(\sigma) \in \mathscr{S}^{M,N}$, $\forall t \geq 0$.*

Proof. As a first step we show that $\Omega^t(\sigma) \in \mathscr{S}$ for every $t \geq 0$. Since $\Omega^t(\sigma)(0) = \sigma(0) > 0$ and $\overline{\Omega^t(\sigma)(k)} = \Omega^t(\sigma)(-k)$, $\Omega^t(\sigma)$ satisfies (S1) and (S2) for every $t \geq 0$. Thus $\Omega^t(\sigma) \in \mathscr{S}$, $\forall t \geq 0$. Next we show that $\Omega^t(\sigma) \in H^M_{\zeta,\mathbb{C}} \cap L^2_N$. Clearly $|\Omega^t(\sigma)(k)| \leq |\sigma(k)|$, thus $\Omega^t(\sigma) \in L^2_N$, $\forall t \geq 0$. Now we show that $\Omega^t(\sigma) \in H^M_{\zeta,\mathbb{C}}$, $\forall t \geq 0$. In particular we prove that $\zeta\partial_k^M \Omega^t(\sigma) \in L^2$, the other cases being analogous. Using the expression (2.104) one gets that

$$\zeta(k)\partial_k^M\Omega^t(\sigma)(k) = e^{-i8k^3t}\left(\zeta(k)\partial_k^M\sigma(k) + \sum_{j=1}^{M-1}\binom{M}{j}\left(-i24tk^2\right)^j\zeta(k)\partial_k^{M-j}\sigma(k) + \left(-i24tk^2\right)^M\zeta(k)\sigma(k)\right) .$$

As $\sigma \in \mathscr{S}^{M,N}$, the first and last term in the r.h.s. above are in L^2. Now we show that for $1 \leq j \leq M-1$, $|k|^{2j}\zeta\partial_k^{M-j}\sigma \in L^2$. We will use the following interpolating estimate, proved in [NP09, Lemma 4]. Assume that $J^a f := (1-\partial_k^2)^{a/2} f \in L^2$ and $\langle k\rangle^b f := (1+|k|^2)^{b/2} f \in L^2$. Then for any $\theta \in (0,1)$

$$\left\| \langle k\rangle^{\theta b} J^{(1-\theta)a} f \right\|_{L^2} \leq c \, \|f\|^{\theta}_{L^2_b} \, \|f\|^{1-\theta}_{H^a_{\mathbb{C}}} \ . \tag{2.105}$$

Note that $\zeta\sigma \in H^M_{\mathbb{C}} \cap L^2_N$, thus we can apply estimate (2.105) with $f = \zeta\sigma$, $b = N$, $a = M$, $\theta = \frac{j}{M}$, to obtain that $\langle k\rangle^{\frac{Nj}{M}}\partial_k^{M-j}(\zeta\sigma) \in L^2$. Since $N \geq 2M$, we have $\langle k\rangle^{2j}\partial_k^{M-j}(\zeta\sigma) \in L^2$. By integration by parts

$$\langle k\rangle^{2j}\zeta(k)\partial_k^{M-j}\sigma(k) = \langle k\rangle^{2j}\partial_k^{M-j}(\zeta\sigma) - \sum_{l=1}^{M-j}\binom{M-j}{l}\langle k\rangle^{2j}\partial_k^l\zeta(k)\,\partial_k^{M-j-l}\sigma(k) \ .$$

Since for any $l \geq 1$ the function $\partial_k^l\zeta$ has compact support, it follows that the r.h.s. above is in L^2. Thus for every $1 \leq j \leq M-1$ we have $\langle k\rangle^{2j}\zeta(k)\partial_k^{M-j}\sigma \in L^2$ and it follows that $\zeta\partial_k^M\Omega^t(\sigma) \in L^2$ for every $t \geq 0$. $\square$

Remark 2.41. *One can adapt the proof above, putting $\zeta(k) \equiv 1$, to shows that the spaces $H^N \cap L^2_M$, with integers $N \geq 2M \geq 2$, are invariant by the Airy flow. Indeed the Fourier transform $\mathcal{F}_-$ conjugates the Airy flow with the linear flow Ω^t, i.e., $U^t_{Airy} = \mathcal{F}_-^{-1} \circ \Omega^t \circ \mathcal{F}_-$.*

Proof of Theorem 2.3. Recall that by [GGKM74] the scattering map S conjugate the KdV flow with the linear flow $\Omega^t(\sigma)(k) := e^{-i8\pi^3k^3t}\sigma(k)$, i.e.,

$$U^t_{KdV} = S^{-1}\circ\Omega^t\circ S \ , \tag{2.106}$$

whereas $U^t_{Airy} = \mathcal{F}_-^{-1}\circ\Omega^t\circ\mathcal{F}_-$. Take now $q\in\mathcal{Q}^{N,M}$, where N, M are integers with $N\geq 2M\geq 8$. By Theorem 2.1, $S(q)\equiv S(q,\cdot)\in\mathscr{S}^{M,N}$. By Lemma 2.40 the flow Ω^t preserves the space $\mathscr{S}^{M,N}$ for every $t\geq 0$. Thus $\Omega^t\circ S(q)\in\mathscr{S}^{M,N}$, $\forall t\geq 0$. By the bijectivity of S it follows that $S^{-1}\circ\Omega^t\circ S(q)\in\mathcal{Q}^{N,M}$ $\forall t\geq 0$. Thus item (i) is proved.

We prove now item (ii). Remark that by item (i), $U^t_{KdV}(q)\in L^2_M$ for any $t\geq 0$. Since U^t_{Airy} preserves the space $H^N\cap L^2_M$ $(N\geq 2M\geq 8)$, it follows that for $q\in\mathcal{Q}^{N,M}$ the difference $U^t_{KdV}(q)-U^t_{Airy}(q)\in H^N\cap L^2_M$, $\forall t\geq 0$. We prove now the smoothing property of the difference $U^t_{KdV}(q)-U^t_{Airy}(q)$. Since $S^{-1}=\mathcal{F}_-^{-1}+B$,

$$U^t_{KdV}(q) = \mathcal{F}_-^{-1}\circ\Omega^t\circ S(q) + B\circ\Omega^t\circ S(q) \tag{2.107}$$

and since $S=\mathcal{F}_-+A$,

$$\mathcal{F}_-^{-1}\circ\Omega^t\circ S(q) = \mathcal{F}_-^{-1}\circ\Omega^t\circ\mathcal{F}_-(q) + \mathcal{F}_-^{-1}\circ\Omega^t\circ A(q) \ .$$

Hence

$$U^t_{KdV}(q) = U^t_{Airy}(q) + \mathcal{F}_-^{-1}\circ\Omega^t\circ A(q) + B\circ\Omega^t\circ S(q). \tag{2.108}$$

The 1-smoothing property of the difference $U^t_{KdV}(q)-U^t_{Airy}(q)$ follows now from the smoothing properties of A and B described in item (ii) of Theorem 2.1. The real analyticity of the map $q\mapsto U^t_{KdV}(q)-U^t_{Airy}(q)$ follows from formula (2.108) and the real analyticity of the maps A, B and S. $\square$

A Auxiliary results.

For the convenience of the reader in this appendix we collect various known estimates used throughout the paper.

Lemma 2.42. *Fix an arbitrary real number c. Then the following holds:*

(A0) The linear map $T_0 : L^2_{1/2,x\geq c}\to L^2_{x\geq c}L^2_{y\geq 0}$ defined by

$$g\mapsto T_0(g)(x,y) := g(x+y) \tag{2.109}$$

is continuous, and there exists a constant $K_c>0$, depending on c, such that

$$\|T_0(g)\|_{L^2_{x\geq c}L^2_{y\geq 0}} \leq K_c\,\|g\|_{L^2_{1/2,x\geq c}} \,. \tag{2.110}$$

(A1) The bilinear map $T_1 : L^2_{x\geq c}\times L^2_{x\geq c}\to L^2_{x\geq c}L^2_{y\geq 0}$ defined by

$$(g,h)\mapsto T_1(g,h)(x,y) := g(x+y)h(x) \tag{2.111}$$

is continuous, and

$$\|T_1(g,h)\|_{L^2_{x\geq c}L^2_{y\geq 0}} \leq \|g\|_{L^2_{x\geq c}}\,\|h\|_{L^2_{x\geq c}} \,. \tag{2.112}$$

(A2) The bilinear map $T_2 : L^2_{x\geq c} \times L^2_{x\geq c}L^2_{y\geq 0} \to L^2_{x\geq c}$ defined by

$$(g,h) \mapsto T_2(g,h)(x) := \int_0^{+\infty} g(x+z)h(x,z)\,dz \tag{2.113}$$

is continuous, and there exists a constant $K_c > 0$, depending on c, such that

$$\|T_2(g,h)\|_{L^2_{x\geq c}} \leq K_c \|g\|_{L^2_{x\geq c}} \|h\|_{L^2_{x\geq c}L^2_{y\geq 0}} . \tag{2.114}$$

(A3) (Hardy inequality) The linear map $T_3 : L^2_{m+1,x\geq c} \to L^2_{m,x\geq c}$ defined by

$$g \mapsto T_3(g)(x) := \int_x^{+\infty} g(z)dz$$

is continuous, and there exists a constant $K_c > 0$, depending on c, such that

$$\|T_3(g)\|_{L^2_{m,x\geq c}} \leq K_c \|g\|_{L^2_{m+1,x\geq c}} \quad .$$

(A4) The bilinear map $T_4 : L^1_{x\geq c} \times L^2_{x\geq c}L^2_{y\geq 0} \to L^2_{x\geq c}L^2_{y\geq 0}$ defined by

$$(g,h) \mapsto T_4(g,h)(x,y) := \int_0^{+\infty} g(x+y+z)h(x,z)dz \tag{2.115}$$

is continuous, and there exists a constant $K_c > 0$, depending on c, such that

$$\|T_4(g,h)\|_{L^2_{x\geq c}L^2_{y\geq 0}} \leq K_c \|g\|_{L^1_{x\geq c}} \|h\|_{L^2_{x\geq c}L^2_{y\geq 0}} . \tag{2.116}$$

(A5) The bilinear map $T_5 : L^2_{x\geq c} \times L^2_{1,x\geq c} \to L^2_{x\geq c}L^2_{y\geq 0}$ defined by

$$(g,h) \mapsto T_5(g,h)(x,y) := \int_0^{+\infty} g(x+y+z)h(x+z)dz \tag{2.117}$$

is bounded and satisfies

$$\|T_5(g,h)\|_{L^2_{x\geq c}L^2_{y\geq 0}} \leq K_c \|g\|_{L^2_{x\geq c}} \|h\|_{L^2_{1,x\geq c}} . \tag{2.118}$$

Proof. Inequality $(A1), (A4)$ can be verified in a straightforward way. To prove $(A0)$ make the change of variable $\xi = x + y$ and remark that

$$\int_c^{+\infty}\int_0^{+\infty} |g(x+y)|^2\,dx\,dy \leq K_c \int_0^{+\infty} |\xi - c|\,|g(\xi)|^2 d\xi \ .$$

We prove now $(A2)$: using Cauchy-Schwartz, one gets

$$\left\| \int_0^{+\infty} g(x+z)h(x,z)\,dz \right\|^2_{L^2_{x\geq c}} \leq \int_c^{+\infty} \left(\int_x^{+\infty} |g(z)|^2\,dz \right) \left(\int_0^{+\infty} |h(x,z)|^2\,dz \right) dx \leq \|g\|^2_{L^2_{x\geq c}} \|h\|^2_{L^2_{x\geq c}L^2_{y\geq 0}} \ .$$

In order to prove $(A3)$ take a function $h \in L^2_{x\geq c}$ and remark that

$$\left| \int_c^{+\infty} dx\, h(x)\, \langle x \rangle^m \int_x^{+\infty} g(z)\,dz \right| = \left| \int_c^{+\infty} dz\, g(z) \int_c^z \langle x \rangle^m h(x) dx \right| \leq \tilde{K}_c \int_c^{+\infty} dz\, \langle z \rangle^m |g(z)| \int_c^z |h(x)|\,dx$$

$$\leq K_c \int_c^{+\infty} dz\, \langle z \rangle^{m+1} |g(z)|\, \frac{\int_c^z |h(x)| dx}{|z-c|} \leq K_c \left\| \langle z \rangle^{m+1} g \right\|_{L^2_{x\geq c}} \|h\|_{L^2_{x\geq c}}$$

where for the last inequality we used the Hardy-Littlewood inequality.
To prove $(A4)$ take a function $f \in L^2_{x\geq c} L^2_{y\geq 0}$, define $\Omega_c = [c,\infty) \times \mathbb{R}^+ \times \mathbb{R}^+$ and remark that

$$\int_{\Omega_c} |g(x+y+z)|\, |h(x,z)|\, |f(x,y)|\, dx\, dy\, dz \leq$$

$$\leq \Big(\int_{\Omega_c} |g(x+y+z)|\, |h(x,z)|^2\, dx\, dy\, dz \Big)^{1/2} \Big(\int_{\Omega_c} |g(x+y+z)|\, |f(x,y)|^2\, dx\, dy\, dz \Big)^{1/2}$$

$$\leq \|g\|_{L^1_{x\geq c}} \|h\|_{L^2_{x\geq c, z\geq 0}} \|f\|_{L^2_{x\geq c} L^2_{y\geq 0}} ,$$

where the first inequality follows by writing $|g| = |g|^{1/2} \cdot |g|^{1/2}$ and applying Cauchy-Schwartz.
To prove $(A5)$ note that

$$\left\| \int_0^{+\infty} g(x+y+z)h(x+z)\,dz \right\|_{L^2_{y\geq 0}} \leq \|g\|_{L^2_{x\geq c}} \int_x^{+\infty} |h(z)|\ dz \ .$$

By $(A3)$ one has that $\left\| \int_x^{+\infty} |h(z)|\ dz \right\|_{L^2_{x\geq c}} \leq K_c \left\| \langle x \rangle h \right\|_{L^2_{x\geq c}}$, then $(A5)$ follows. □

B Analytic maps in complex Banach spaces

In this appendix we recall the definition of an analytic map from [Muj86].

Let E and F be complex Banach spaces. A map $\tilde{P}^k : E^k \to F$ is said to be k-multilinear if $\tilde{P}^k(u^1, \ldots, u^k)$ is linear in each variable u^j; a multilinear map is said to be bounded if there exist a constant C such that

$$\left\| \tilde{P}^k(u^1, \cdots, u^k) \right\| \leq C \left\| u^1 \right\| \cdots \left\| u^k \right\| \quad \forall u^1, \ldots, u^k \in E.$$

Its norm is defined by

$$\left\|\tilde{P}^k\right\| := \sup_{u^j \in E,\ \|u^j\| \leq 1} \left\|\tilde{P}^k(u^1, \dots, u^k)\right\|.$$

A map $P^k : E \to F$ is said to be a polynomial of order k if there exists a k-multilinear map $\tilde{P}^k : E \to F$ such that

$$P^k(u) = \tilde{P}^k(u, \dots, u) \quad \forall u \in E.$$

The polynomial is bounded if it has finite norm

$$\left\|P^k\right\| := \sup_{\|u\| \leq 1} \left\|P^k(u)\right\|.$$

We denote with $\mathcal{P}^k(E,F)$ the vector space of all bounded polynomials of order k from E into F.

Definition 2.43. *Let E and F be complex Banach spaces. Let U be a open subset of E. A mapping $f : U \to F$ is said to be analytic if for each $a \in U$ there exists a ball $B_r(a) \subset U$ with center a and radius $r > 0$ and a sequence of polynomials $P^k \in \mathcal{P}^k(E,F)$, $k \geq 0$, such that*

$$f(u) = \sum_{k=0}^{\infty} P^k(u-a)$$

is convergent uniformly for $u \in B_r(a)$; i.e., for any $\epsilon > 0$ there exists $K > 0$ so that

$$\left\| f(u) - \sum_{k=0}^{K} P^k(u-a) \right\| \leq \epsilon$$

for any $u \in B_r(a)$.

Finally let us recall the notion of real analytic map.

Definition 2.44. *Let E, F be real Banach spaces and denote by $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$ their complexifications. Let $U \subset E$ be open. A map $f : U \to F$ is called real analytic on U if for each point $u \in U$ there exists a neighborhood V of u in $E_{\mathbb{C}}$ and an analytic map $g : V \to F_{\mathbb{C}}$ such that $f = g$ on $U \cap V$.*

Remark 2.45. *The notion of an analytic map in Definition 2.43 is equivalent to the notion of a $\mathbb{C}$-differentiable map. Recall that a map $f : U \to F$, where U, E and F are given as in Definition 2.43, is said to be $\mathbb{C}$-differentiable if for each point $a \in U$ there exists a linear, bounded operator $A : E \to F$ such that*

$$\lim_{u \to a} \frac{\|f(u) - f(a) - A(u-a)\|_F}{\|u-a\|_E} = 0.$$

Therefore analytic maps inherit the properties of $\mathbb{C}$-differentiable maps; in particular the composition of analytic maps is analytic. For a proof of the equivalence of the two notions see [Muj86], Theorem 14.7.

Remark 2.46. *Any $P^k \in \mathcal{P}^k(E,F)$ is an analytic map. Let $f(u) = \sum_{m=0}^{\infty} P^m(u)$ be a power series from E into F with infinite radius of convergence with $P^m \in \mathcal{P}^m(E,F)$. Then f is analytic ([Muj86], example 5.3, 5.4).*

C Properties of the solutions of integral equation (2.93)

In this section we discuss some properties of the solution of equation (2.93) which we rewrite as

$$g(x,y) + \int_0^{+\infty} F_\sigma(x+y+z)\, g(x,z)\, dz = h_\sigma(x,y) \ . \tag{2.119}$$

Here $\sigma \in \mathscr{S}^{4,N}$, $N \geq 0$, h_σ is a function $h_\sigma : [c, +\infty) \times [0, +\infty) \to \mathbb{R}$, with c arbitrary, which satisfies (P). We denote by

$$\|h\|_0 := \|h\|_{L^2_{x\geq c} L^2_{y \geq 0}} + \|h(\cdot, 0)\|_{L^2_{x\geq c}} \ . \tag{2.120}$$

Furthermore $F_\sigma : \mathbb{R} \to \mathbb{R}$ is a function that satisfies

(H) The map $\sigma \mapsto F_\sigma$ $[\sigma \mapsto F'_\sigma]$ is real analytic as a map from $\mathscr{S}^{4,N}$ to $H^1 \cap L^2_3$ $[L^2_4]$. Moreover the operators $Id \pm \mathcal{K}_{x,\sigma} : L^2_{y\geq 0} \to L^2_{y \geq 0}$ with $\mathcal{K}_{x,\sigma}$ defined as

$$\mathcal{K}_{x,\sigma}[f](y) := \int_0^{+\infty} F_\sigma(x+y+z)\, f(z)\, dz \tag{2.121}$$

are invertible for any $x \geq c$, and there exists a constant $C_\sigma > 0$, depending locally uniformly on $\sigma \in H^4_{\zeta, \mathbb{C}} \cap L^2_N$, such that

$$\sup_{x \geq c} \left\| (Id \pm \mathcal{K}_{x,\sigma})^{-1} \right\|_{\mathcal{L}(L^2_{y\geq 0})} \leq C_\sigma \ . \tag{2.122}$$

Finally $\sigma \mapsto (Id \pm \mathcal{K}_{x,\sigma})^{-1}$ are real analytic as maps from $\mathscr{S}^{4,N}$ to $\mathcal{L}(L^2_{x\geq c} L^2_{y \geq 0})$.

Remark 2.47. *The pairing*

$$\mathcal{L}(L^2_{x\geq c} L^2_{y\geq 0}) \times L^2_{x \geq c} L^2_{y\geq 0} \to L^2_{x\geq c} L^2_{y \geq 0}, \qquad (H, f) \mapsto H[f]$$

is a bounded bilinear map and hence analytic. Let now $\sigma \mapsto h_\sigma$ be a real analytic map from $\mathscr{S}^{4,0}$ to $L^2_{x\geq c} L^2_{y\geq 0}$ and let $\mathcal{K}_\sigma$ as in (H). Then by Lemma 2.54 (iii) it follows that $\sigma \mapsto (Id + \mathcal{K}_\sigma)^{-1}[h_\sigma]$ is real analytic as a map from $\mathscr{S}^{4,0}$ to $L^2_{x\geq c} L^2_{y \geq 0}$ as well.

Remark 2.48. *By the Sobolev embedding theorem, assumption (H) implies that $F_\sigma \in C^{0,\gamma}(\mathbb{R}, \mathbb{C})$, $\gamma < \frac{1}{2}$.*

By assumption (H) the map $(c, \infty) \to \mathcal{L}(L^2_{y\geq 0})$, $x \mapsto \mathcal{K}_{x,\sigma}$ is differentiable and its derivative is the operator

$$\mathcal{K}'_{x,\sigma}[f](y) = \int_0^{+\infty} F'_\sigma(x+y+z)\, f(z)\, dz \ , \tag{2.123}$$

as one verifies using that for $x > c$ and $\epsilon \neq 0$ sufficiently small

$$\begin{aligned} \left\| \frac{\mathcal{K}_{x+\epsilon,\sigma} - \mathcal{K}_{x,\sigma}}{\epsilon} - \mathcal{K}'_{x,\sigma} \right\|_{\mathcal{L}(L^2_{y\geq 0})} &\leq \int_x^{+\infty} \left| \frac{F_\sigma(z+\epsilon) - F_\sigma(z)}{\epsilon} - F'_\sigma(z) \right| dz \\ \leq \frac{1}{|\epsilon|} \left| \int_0^{\epsilon} \int_x^{+\infty} |F'_\sigma(z+s) - F'_\sigma(z)|\, dz\, ds \right| &\leq \sup_{|s|\leq|\epsilon|} \int_x^{+\infty} |F'_\sigma(z+s) - F'_\sigma(z)|\, dz \end{aligned} \tag{2.124}$$

and the fact that the translations are continuous in L^1. Therefore the following lemma holds

Lemma 2.49. *$\mathcal{K}_{x,\sigma}$ and thus $(Id+\mathcal{K}_{x,\sigma})^{-1}$ is a family of operators from $L^2_{y\geq 0}$ to $L^2_{y\geq 0}$ which depends continuously on the parameter x. Moreover the map $(c,\infty)\to\mathcal{L}(L^2_{y\geq 0})$, $x\mapsto\mathcal{K}_{x,\sigma}$ is differentiable and its derivative is the operator $\mathcal{K}'_{x,\sigma}$ defined in (2.123).*

Lemma 2.50. *Let F_σ satisfy assumption (H), and $g_\sigma\in C^0_{x\geq c}L^2_{y\geq 0}\cap L^2_{x\geq c}L^2_{y\geq 0}$ be such that $\|g_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}}\leq K_c\,\|\sigma\|^2_{H^4_{\zeta,\mathbb{C}}\cap L^2_N}$ and $\mathscr{S}^{4,N}\to L^2_{x\geq c}L^2_{y\geq 0}$, $\sigma\mapsto g_\sigma$ be real analytic. Then*

$$\mathbf{F}_R(x,y):=\int_0^{+\infty}F_\sigma(x+y+z)\,g_\sigma(x,z)\,dz$$

satisfies (P).

Proof. $(P1)$ For $\epsilon\neq 0$ sufficiently small

$$\begin{aligned}\|\mathbf{F}_R(x+\epsilon,\cdot)-\mathbf{F}_R(x,\cdot)\|_{L^2_{y\geq 0}}\leq&\ \|F_\sigma(x+\epsilon,\cdot)-F_\sigma(x,\cdot)\|_{L^1}\,\|g_\sigma(x+\epsilon,\cdot)\|_{L^2_{y\geq 0}}\\ &+\|F_\sigma\|_{L^1}\,\|g_\sigma(x+\epsilon,\cdot)-g_\sigma(x,\cdot)\|_{L^2_{y\geq 0}}\end{aligned}$$

which goes to 0 as $\epsilon\to 0$, proving that $\mathbf{F}_R\in C^0_{x\geq c}L^2_{y\geq 0}$. Furthermore, by Lemma 2.42 $(A4)$, $\mathbf{F}_R\in L^2_{x\geq c}L^2_{y\geq 0}$ and fulfills

$$\|\mathbf{F}_R\|_{L^2_{x\geq c}L^2_{y\geq 0}}\leq\|F_\sigma\|_{L^1}\,\|g_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}}\leq K_c\,\|\sigma\|^2_{H^4_{\zeta,\mathbb{C}}\cap L^2_N}\ . \tag{2.125}$$

Now we show that $\mathbf{F}_R\in C^0_{x\geq c,y\geq 0}$. Let $(x_n)_{n\geq 1}\subseteq[c,\infty)$ and $(y_n)_{n\geq 1}\subseteq[0,\infty)$ be two sequences such that $x_n\to x_0$, $y_n\to y_0$. Then $F_\sigma(x_n+y_n+\cdot)g_\sigma(x_n,\cdot)\to F_\sigma(x_0+y_0+\cdot)g_\sigma(x_0,\cdot)$ in $L^1_{z\geq 0}$ as $n\to\infty$. Indeed

$$\begin{aligned}&\|F_\sigma(x_n+y_n+\cdot)g_\sigma(x_n,\cdot)-F_\sigma(x_0+y_0+\cdot)g_\sigma(x_0,\cdot)\|_{L^1_{z\geq 0}}\leq\\ &\quad\leq\|F_\sigma(x_n+y_n+\cdot)-F_\sigma(x_0+y_0+\cdot)\|_{L^2_{z\geq 0}}\,\|g_\sigma(x_n,\cdot)\|_{L^2_{y\geq 0}}\\ &\quad+\|F_\sigma(x_0+y_0+\cdot)\|_{L^2_{z\geq 0}}\,\|g_\sigma(x_n,\cdot)-g_\sigma(x_0,\cdot)\|_{L^2_{y\geq 0}}\,,\end{aligned}$$

and the r.h.s. of the inequality above goes to 0 as $(x_n,y_n)\to(x_0,y_0)$, by the continuity of the translations in L^2 and the fact that $g_\sigma\in C^0_{x\geq c}L^2_{y\geq 0}$. Thus it follows that $\mathbf{F}_R(x_n,y_n)\to\mathbf{F}_R(x_0,y_0)$ as $n\to\infty$, i.e., $\mathbf{F}_R\in C^0_{x\geq c,y\geq 0}$.
We evaluate $\mathbf{F}_R$ at $y=0$, getting

$$\mathbf{F}_R(x,0)=\int_0^{+\infty}F_\sigma(x+z)g_\sigma(x,z)\,dz\ .$$

By Lemma 2.42 $(A2)$, $\mathbf{F}_R(\cdot,0)\in L^2_{x\geq c}$ and fulfills

$$\|\mathbf{F}_R(\cdot,0)\|_{L^2_{x\geq c}}\leq\|F_\sigma\|_{L^2}\,\|g_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}}\leq K_c\,\|\sigma\|^2_{H^4_{\zeta,\mathbb{C}}\cap L^2_N}\ . \tag{2.126}$$

$(P2)$ It follows from (2.125) and (2.126).

$(P3)$ It follows by Lemma 2.42 $(A2)$ and the fact that $\mathbf{F}_R$ and $\mathbf{F}_R(\cdot,0)$ are composition of real analytic maps. □

We study now the solution of equation (2.119).

Lemma 2.51. *Assume that h_σ satisfies (P) and F_σ satisfies (H). Then equation* (2.119) *has a unique solution g_σ in $C^0_{x\geq c}L^2_{y\geq 0}\cap L^2_{x\geq c}L^2_{y\geq 0}$ which satisfies (P).*

Proof. We start to show that g_σ exists and satisfies $(P1)$. Since h_σ satisfies (P) and F_σ satisfies (H), it follows that for any $x\geq c$, $g_\sigma(x,\cdot):=(Id+\mathcal{K}_{x,\sigma})^{-1}[h_\sigma(x,\cdot)]$ is the unique solution in $L^2_{y\geq 0}$ of the integral equation (2.119). Furthermore, by (2.122), $\|g_\sigma(x,\cdot)\|_{L^2_{y\geq 0}}\leq C_\sigma\|h_\sigma(x,\cdot)\|_{L^2_{y\geq 0}}$, which implies

$$\|g_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}}\leq C_\sigma\|h_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}} \ . \tag{2.127}$$

Since $h_\sigma\in C^0_{x\geq c}L^2_{y\geq 0}$, Lemma 2.49 implies that $g_\sigma\in C^0_{x\geq c}L^2_{y\geq 0}$ as well. Thus we have proved that $g_\sigma\in C^0_{x\geq c}L^2_{y\geq 0}\cap L^2_{x\geq c}L^2_{y\geq 0}$. Now write

$$g_\sigma(x,y)=h_\sigma(x,y)-\int_0^{+\infty}F_\sigma(x+y+z)g_\sigma(x,z)\,dz \ . \tag{2.128}$$

By Lemma 2.50 and the assumption that h_σ satisfies (P), it follows that the r.h.s. of formula (2.128) satisfies (P). $\square$

The following lemma will be useful in the following:

Lemma 2.52. *(i) Let F_σ satisfy (H), and $g_\sigma\in C^0_{x\geq c}L^2_{y\geq 0}\cap L^2_{x\geq c}L^2_{y\geq 0}$ be such that $\|g_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}}\leq K_c\|\sigma\|^2_{H^4_{\zeta,\mathbb{C}}\cap L^2_N}$ and $\mathscr{S}^{4,N}\to L^2_{x\geq c}L^2_{y\geq 0}$, $\sigma\mapsto g_\sigma$ be real analytic. Denote*

$$\Phi_\sigma(x,y):=\int_0^{+\infty}F'_\sigma(x+y+z)g_\sigma(x,z)\,dz \ . \tag{2.129}$$

Then $\Phi_\sigma\in C^0_{x\geq c}L^2_{y\geq 0}\cap L^2_{x\geq c}L^2_{y\geq 0}$, the map $\mathscr{S}^{4,N}\to L^2_{x\geq c}L^2_{y\geq 0}$, $\sigma\mapsto\Phi_\sigma$ is real analytic and

$$\|\Phi_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}}\leq K_c\|\sigma\|^2_{H^4_{\zeta,\mathbb{C}}\cap L^2_N} \ , \tag{2.130}$$

where $K_c>0$ depends locally uniformly on $\sigma\in H^4_{\zeta,\mathbb{C}}\cap L^2_N$.

(ii) Let g_σ as in item (i), and furthermore let $g_\sigma\in C^0_{x\geq c,y\geq 0}$ and $g_\sigma(\cdot,0)\in L^2_{x\geq c}$. Assume furthermore that $\partial_y g_\sigma$ satisfies the same assumptions as g_σ in item (i). Then Φ_σ, defined in (2.129)*, satisfies (P).*

(iii) Assume that F_σ satisfies (H) and that the map $\mathscr{S}^{4,N}\to H^1_{x\geq c}$, $\sigma\mapsto b_\sigma$ is real analytic with $\|b_\sigma\|_{H^1_{x\geq c}}\leq K_c\|\sigma\|^2_{H^4_{\zeta,\mathbb{C}}\cap L^2_N}$. Then the function

$$\phi_\sigma(x,y):=F_\sigma(x+y)b_\sigma(x)$$

satisfies (P).

Proof. (i) Clearly $\|\Phi_\sigma(x,\cdot)\|_{L^2_{y\geq 0}} \leq \|F'_\sigma\|_{L^1} \|g_\sigma(x,\cdot)\|_{L^2_{y\geq 0}}$, and since $g_\sigma \in L^2_{x\geq c}L^2_{y\geq 0}$ it follows that $\Phi_\sigma \in L^2_{x\geq c}L^2_{y\geq 0}$ with $\|\Phi_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}} \leq \|F'_\sigma\|_{L^2_4} \|g_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}}$, which implies (2.129). We show now that $\Phi_\sigma \in C^0_{x\geq c}L^2_{y\geq 0}$. For $\epsilon \neq 0$ one has

$$\|\Phi_\sigma(x+\epsilon,\cdot) - \Phi_\sigma(x,\cdot)\|_{L^2_{y\geq 0}} \leq \|F'_\sigma(\cdot+\epsilon) - F'_\sigma\|_{L^1} \|g_\sigma(x,\cdot)\|_{L^2_{y\geq 0}} + \|F'_\sigma\|_{L^1} \|g_\sigma(x+\epsilon,\cdot) - g_\sigma(x,\cdot)\|_{L^2_{y\geq 0}} \ .$$

The continuity of the translation in L^1 and the assumption $g_\sigma \in C^0_{x\geq c}L^2_{y\geq 0}$ imply that $\|\Phi_\sigma(x+\epsilon,\cdot) - \Phi_\sigma(x,\cdot)\|_{L^2_{y\geq 0}} \to 0$ as $\epsilon \to 0$, thus $\Phi_\sigma \in C^0_{x\geq c}L^2_{y\geq 0}$. The real analyticity of $\sigma \mapsto \Phi_\sigma$ follows from Lemma 2.42 $(A4)$ and the fact that Φ_σ is composition of real analytic maps.

(ii) Fix $x \geq c$ and use integration by parts to write

$$\Phi_\sigma(x,y) = -F_\sigma(x+y)g_\sigma(x,0) - \int_0^{+\infty} F_\sigma(x+y+z)\partial_z g_\sigma(x,z)\, dz \ , \tag{2.131}$$

where we used that since $F_\sigma \in H^1$ $[g(x,\cdot) \in H^1_{y\geq 0}]$, $\lim_{x\to\infty} F_\sigma(x) = 0$ $[\lim_{y\to\infty} g_\sigma(x,y) = 0]$. By the assumption and the proof of Lemma 2.50 $(P1)$, $\Phi_\sigma \in C^0_{x\geq c, y\geq 0}$. We evaluate (2.131) at $y=0$ to get the formula

$$\Phi_\sigma(x,0) = -F_\sigma(x)g_\sigma(x,0) - \int_0^{+\infty} F_\sigma(x+z)\partial_z g_\sigma(x,z)\, dz \ .$$

Together with Lemma 2.42 $(A2)$ we have the estimate

$$\|\Phi_\sigma(\cdot,0)\|_{L^2_{x\geq c}} \leq \|F_\sigma\|_{H^1} \left(\|g_\sigma(\cdot,0)\|_{L^2_{x\geq c}} + \|\partial_y g_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}} \right) \leq K_c \|\sigma\|^2_{H^4_{\zeta,\mathbb{C}} \cap L^2_N} \ . \tag{2.132}$$

Estimate (2.132) together with estimate (2.130) imply that Φ_σ satisfies $(P2)$. Finally $\sigma \mapsto \Phi_\sigma(\cdot,0)$ is real analytic, being a composition of real analytic maps.

(iii) We skip an easy proof. □

If the function h_σ is more regular one deduces better regularity properties of the corresponding solution of (2.119).

Lemma 2.53. *Consider the integral equation* (2.119) *and assume that F_σ satisfies (H). Assume that h_σ, $\partial_x h_\sigma$, $\partial_y h_\sigma$ satisfy (P). Then g_σ solution of* (2.119) *satisfies (P). Its derivatives $\partial_x g_\sigma$ and $\partial_y g_\sigma$ satisfy (P) and solve the equations*

$$(Id + \mathcal{K}_{x,\sigma})\,[\partial_x g_\sigma] = \partial_x h_\sigma - \mathcal{K}'_{x,\sigma}[g_\sigma] \ , \tag{2.133}$$

$$\partial_y g_\sigma = \partial_y h_\sigma - \mathcal{K}'_{x,\sigma}[g_\sigma] \ . \tag{2.134}$$

Proof. By Lemma 2.51, g_σ satisfies (P).

$\partial_y g_\sigma$ satisfies (P). For $\epsilon \neq 0$ sufficiently small, we have in $L^2_{y\geq 0}$

$$\frac{g_\sigma(x,y+\epsilon) - g_\sigma(x,y)}{\epsilon} = \Psi^\epsilon_\sigma(x,y)$$

where

$$\Psi_\sigma^\epsilon(x,y) := \frac{h_\sigma(x,y+\epsilon)-h_\sigma(x,y)}{\epsilon} - \int_0^{+\infty} \frac{F_\sigma(x+y+\epsilon+z)-F_\sigma(x+y+z)}{\epsilon} g_\sigma(x,z)\,dz \ . \quad (2.135)$$

Define

$$\Psi_\sigma^0(x,y) := \partial_y h_\sigma(x,y) - \int_0^{+\infty} F_\sigma'(x+y+z) g_\sigma(x,z)\,dz \ .$$

Since $\partial_y h_\sigma$ and g_σ satisfy (P), by Lemma 2.52 (i) it follows that $\Psi_\sigma^0 \in C^0_{x\geq c} L^2_{y\geq 0} \cap L^2_{x\geq c} L^2_{y\geq 0}$, the map $\mathscr{S}^{4,N} \to L^2_{x\geq c} L^2_{y\geq 0}$, $\sigma \mapsto \Psi_\sigma^0$ is real analytic and $\left\| \Psi_\sigma^0 \right\|_{L^2_{x\geq c} L^2_{y\geq 0}} \leq K_c \left\| \sigma \right\|^2_{H^4_{\zeta,\mathbb{C}} \cap L^2_N}$. Furthermore one verifies that

$$\partial_y g_\sigma(x,\cdot) = \lim_{\epsilon\to 0} \frac{g_\sigma(x,\cdot+\epsilon)-g_\sigma(x,\cdot)}{\epsilon} = \lim_{\epsilon\to 0} \Psi_\sigma^\epsilon(x,\cdot) = \Psi_\sigma^0(x,\cdot) \qquad \text{in } L^2_{y\geq 0} \ .$$

Thus $\partial_y g_\sigma$ fulfills

$$\partial_y g_\sigma(x,y) = \partial_y h_\sigma(x,y) - \int_0^{+\infty} F_\sigma'(x+y+z) g_\sigma(x,z) dz \ , \quad (2.136)$$

i.e., $\partial_y g_\sigma$ satisfies equation (2.134). Since $\partial_y g_\sigma = \Psi_\sigma^0$, g_σ satisfies the assumptions of Lemma 2.52 (ii). Since $\partial_y h_\sigma$ satisfies (P) as well, it follows that $\partial_y g_\sigma$ satisfies (P).

$\partial_x g_\sigma$ *satisfies* (P). For $\epsilon \neq 0$ small enough we have in $L^2_{y\geq 0}$

$$(Id + \mathcal{K}_{x+\epsilon,\sigma}) \left[\frac{g_\sigma(x+\epsilon,\cdot)-g_\sigma(x,\cdot)}{\epsilon} \right] = \Phi_\sigma^\epsilon(x,\cdot)$$

where

$$\Phi_\sigma^\epsilon(x,y) := \frac{h_\sigma(x+\epsilon,y)-h_\sigma(x,y)}{\epsilon} - \int_0^{+\infty} \frac{F_\sigma(x+y+\epsilon+z)-F_\sigma(x+y+z)}{\epsilon} g_\sigma(x,z)\,dz \ .$$

Define

$$\Phi_\sigma^0(x,y) := \partial_x h_\sigma(x,y) - \int_0^{+\infty} F_\sigma'(x+y+z) g_\sigma(x,z)\,dz \ .$$

Proceeding as above, one proves that Φ_σ^0 satisfies (P), and

$$\lim_{\epsilon\to 0} \Phi_\sigma^\epsilon(x,\cdot) = \Phi_\sigma^0(x,\cdot) \qquad \text{in } L^2_{y\geq 0} \ .$$

Together with Lemma 2.49 we get for $x > c$ in $L^2_{y\geq 0}$

$$\partial_x g_\sigma(x,\cdot) = \lim_{\epsilon\to 0} \frac{g_\sigma(x+\epsilon,\cdot)-g_\sigma(x,\cdot)}{\epsilon} = \lim_{\epsilon\to 0} \left(Id + \mathcal{K}_{x+\epsilon,\sigma}\right)^{-1} \Phi_\sigma^\epsilon(x,\cdot) = \left(Id + \mathcal{K}_{x,\sigma}\right)^{-1} \Phi_\sigma^0(x,\cdot) \ . \quad (2.137)$$

In particular $(Id+\mathcal{K}_\sigma)(\partial_x g_\sigma(x,\cdot)) = \Phi^0_\sigma(x,\cdot)$. Since Φ^0_σ satisfies (P), by Lemma 2.51, $\partial_x g_\sigma$ satisfies (P). Formula (2.137) implies that

$$\partial_x g_\sigma(x,y) + \int_0^{+\infty} F_\sigma(x+y+z)\partial_x g_\sigma(x,z)\,dz = \partial_x h_\sigma(x,y) - \int_0^{+\infty} F'_\sigma(x+y+z) g_\sigma(x,z)\,dz\ , \quad (2.138)$$

namely $\partial_x g_\sigma$ satisfies equation (2.133).

□

D Proof from Section 4

D.1 Properties of $\mathcal{K}^\pm_{x,\sigma}$ and $f_{\pm,\sigma}$.

We begin with proving some properties of $\mathcal{K}^\pm_{x,\sigma}$ and $f_{\pm,\sigma}$, defined in (2.94) and (2.96), which will be needed later.

Properties of $Id+\mathcal{K}^\pm_{x,\sigma}$. In order to solve the integral equations (2.93) we need the operator $Id+\mathcal{K}^+_{x,\sigma}$ to be invertible on $L^2_{y\geq 0}$ (respectively $Id+\mathcal{K}^-_{x,\sigma}$ to be invertible on $L^2_{y\leq 0}$). The following result is well known:

Lemma 2.54 ([DT79, CK87a]). *Let $\sigma \in \mathscr{S}^{4,0}$ and fix $c \in \mathbb{R}$. Then the following holds:*

(i) For every $x \geq c$, $\mathcal{K}^+_{x,\sigma} : L^2_{y\geq 0} \to L^2_{y\geq 0}$ is a bounded linear operator; moreover

$$\sup_{x\geq c} \left\|\mathcal{K}^+_{x,\sigma}\right\|_{\mathcal{L}(L^2_{y\geq 0})} < 1, \quad \text{and} \quad \left\|\mathcal{K}^+_{x,\sigma}\right\|_{\mathcal{L}(L^2_{y\geq 0})} \leq \int_x^{+\infty} |F_{+,\sigma}(\xi)|\,d\xi \to 0 \quad \text{if} \quad x \to +\infty. \tag{2.139}$$

(ii) The map $\mathcal{K}^+_\sigma : L^2_{x\geq c}L^2_{y\geq 0} \to L^2_{x\geq c}L^2_{y\geq 0}$, $f \mapsto \mathcal{K}^+_\sigma[f]$, where $\mathcal{K}^+_\sigma[f](x,y) := K^+_{x,\sigma}[f](y)$, is linear and bounded. Moreover the operators $Id \pm \mathcal{K}^+_\sigma$ are invertible on $L^2_{x\geq c}L^2_{y\geq 0}$ and there exists a constant $K_c > 0$, which depends locally uniformly on $\sigma \in \mathscr{S}^{4,0}$, such that

$$\left\|\left(Id \pm \mathcal{K}^+_\sigma\right)^{-1}\right\|_{\mathcal{L}(L^2_{x\geq c}L^2_{y\geq 0})} \leq K_c. \tag{2.140}$$

(iii) $\sigma \mapsto (Id \pm \mathcal{K}^+_\sigma)^{-1}$ are real analytic as maps from $\mathscr{S}^{4,0}$ to $\mathcal{L}(L^2_{x\geq c}L^2_{y\geq 0})$.

Analogous results hold also for $\mathcal{K}^-_{x,\sigma}$ replacing $L^2_{x\geq c}L^2_{y\geq 0}$ by $L^2_{x\leq c}L^2_{y\leq 0}$.

Properties of $f_{\pm,\sigma}$. First note that $f_{\pm,\sigma}$, defined by (2.96), are well defined. Indeed for any $\sigma \in \mathscr{S}^{4,0}$, Proposition 2.33 implies that $F_{\pm,\sigma} \in H^1 \cap L^2_3 \subset L^2$. Hence for any $x \geq c,\ y \geq 0$ the map given by $z \mapsto F_{+,\sigma}(x+y+z)F_{+,\sigma}(x+z)$ is in $L^1_{z\geq 0}$. Similarly, for any $x \geq c,\ y \geq 0$, the map given by $z \mapsto F_{-,\sigma}(x+y+z)F_{-,\sigma}(x+z)$ is in $L^1_{z\leq 0}$.

In the following we will use repeatedly the Hardy inequality [HLP88]

$$\left\| \langle x \rangle^m \int_x^{+\infty} g(z)dz \right\|_{L^2_{x\geq c}} \leq K_c \left\| \langle x \rangle^{m+1} g \right\|_{L^2_{x\geq c}}, \qquad \forall\, m \geq 0 \ . \tag{2.141}$$

The inequality is well known, but for sake of completeness we give a proof of it in Lemma 2.42 $(A3)$.

We analyze now the maps $\sigma \mapsto f_{\pm,\sigma}$. Since the analysis of $f_{+,\sigma}$ and the one of $f_{-,\sigma}$ are similar, we will consider $f_{+,\sigma}$ only. To shorten the notation we will suppress the subscript " $+$ " in what follows.

Lemma 2.55. *Fix $N \in \mathbb{Z}_{\geq 0}$ and let $\sigma \in \mathscr{S}^{4,N}$. Let $f_\sigma \equiv f_{+,\sigma}$ be given as in* (2.96). *Then for every $j_1, j_2 \in \mathbb{Z}_{\geq 0}$ with $0 \leq j_1 + j_2 \leq N+1$, the function $\partial_x^{j_1}\partial_y^{j_2} f_\sigma$ satisfies* (P).

Proof. We prove at the same time $(P1), (P2)$ and $(P3)$ for any $j_1, j_2 \geq 0$ with $j_1 + j_2 = n$ for any $0 \leq n \leq N+1$.

Case $n = 0$. Then $j_1 = j_2 = 0$. By Proposition 2.33, for any $N \in \mathbb{Z}_{\geq 0}$ one has $F_\sigma \equiv F_{+,\sigma} \in H^1 \cap L^2_3$.

$(P1)$ We show that $f_\sigma \in C^0_{x\geq c} L^2_{y\geq 0}$. For any $x \geq c$ fixed one has $\|f_\sigma(x,\cdot)\|_{L^2_{y\geq 0}} \leq \|F_\sigma\|_{L^1} \|F_\sigma(x+\cdot)\|_{L^2_{y\geq 0}}$, which shows that $f_\sigma(x,\cdot) \in L^2_{y\geq 0}$. For $\epsilon \neq 0$ sufficiently small one has

$$\begin{aligned} \|f_\sigma(x+\epsilon,\cdot) - f_\sigma(x,\cdot)\|_{L^2_{x\geq c}} &\leq \|F_\sigma\|_{L^1} \|F_\sigma(x+\epsilon+\cdot) - F_\sigma(x+\cdot)\|_{L^2_{y\geq 0}} \\ &\quad + \|F_\sigma(\epsilon+\cdot) - F_\sigma\|_{L^1} \|F_\sigma(x+\cdot)\|_{L^2_{y\geq 0}} \end{aligned}$$

which goes to 0 as $\epsilon \to 0$, due to the continuity of the translations in L^p-space, $1 \leq p < \infty$. Thus $f_\sigma \in C^0_{x\geq c} L^2_{y\geq 0}$.

We show now that $f_\sigma \in L^2_{x\geq c} L^2_{y\geq 0}$. Introduce $h_\sigma(x,y) := F_\sigma(x+y)$. Then $h_\sigma \in L^2_{x\geq c} L^2_{y\geq 0}$, since for some $C, C' > 0$

$$\|h_\sigma\|_{L^2_{x\geq c} L^2_{y\geq 0}} \leq C \|F_\sigma\|_{L^2_{1/2, x\geq c}} \leq C' \|\sigma\|_{H^4_{\zeta,\mathbb{C}}} \tag{2.142}$$

where for the first [second] inequality we used Lemma 2.42 $(A0)$ [Proposition 2.33 (i)]. By Lemma 2.42$(A4)$ and using once more Proposition 2.33 (i), one gets

$$\|f_\sigma\|_{L^2_{x\geq c} L^2_{y\geq 0}} \leq C'' \|F_\sigma\|_{L^1_{x\geq c}} \|h_\sigma\|_{L^2_{x\geq c} L^2_{y\geq 0}} \leq C''' \|F_\sigma\|_{L^2_1} \|h_\sigma\|_{L^2_{x\geq c} L^2_{y\geq 0}} \leq C'''' \|\sigma\|^2_{H^4_{\zeta,\mathbb{C}}} \ , \tag{2.143}$$

for some $C'', C''', C'''' > 0$. Thus $f_\sigma \in L^2_{x\geq c} L^2_{y\geq 0}$.

To show that $f_\sigma \in C^0_{x\geq c, y\geq 0}$ proceed as in Lemma 2.50.

Finally we show that $f_\sigma(\cdot,0) \in L^2_{x\geq c}$. Evaluate (2.96) at $y = 0$ to get $f_\sigma(x,0) = \int_x^{+\infty} F_\sigma^2(z)\,dz$. Using the Hardy inequality (2.141), $F_\sigma(x) = -\int_x^{+\infty} F'_\sigma(s)\,ds$ and Proposition 2.33 one obtains

$$\begin{aligned} \|f_\sigma(\cdot,0)\|_{L^2_{x\geq c}} &\leq \left\|\langle x\rangle F_\sigma^2\right\|_{L^2_{x\geq c}} \leq \|\langle x\rangle F_\sigma\|_{L^\infty_{x\geq c}} \|F_\sigma\|_{L^2_{x\geq c}} \leq K_c \|\langle x\rangle F'_\sigma\|_{L^1_{x\geq c}} \|F_\sigma\|_{L^2_{x\geq c}} \\ &\leq K'_c \left\|\langle x\rangle^2 F'_\sigma\right\|_{L^2_{x\geq c}} \|F_\sigma\|_{L^2_{x\geq c}} \leq K''_c \|\sigma\|^2_{H^4_{\zeta,\mathbb{C}} \cap L^2_N} \ , \end{aligned} \tag{2.144}$$

for some constants $K_c, K'_c, K''_c > 0$. Thus $f_\sigma(\cdot,0) \in L^2_{x\geq c}$.

($P2$) It follows from (2.143) and (2.144).

($P3$) By Proposition 2.33 (i), $\mathscr{S}^{4,0} \to H^1_{\mathbb{C}} \cap L^2_3$, $\sigma \mapsto F_\sigma$ is real analytic and by Lemma 2.42 ($A0$) so is $\mathscr{S}^{4,0} \to L^2_{x\geq c} L^2_{y\geq 0}$, $\sigma \mapsto h_\sigma$. By Lemma 2.42 ($A4$) it follows that $\mathscr{S}^{4,0} \to L^2_{x\geq c} L^2_{y\geq 0}$, $\sigma \mapsto f_\sigma$ is real analytic. Since the map $\sigma \mapsto f_\sigma(\cdot, 0)$ is a composition of real analytic maps, it is real analytic as a map from $\mathscr{S}^{4,N}$ to $L^2_{x\geq c}$.

Case $n \geq 1$. By Proposition 2.33, $F_\sigma \in H^{N+1}$ and $\|F_\sigma\|_{H^{N+1}} \leq C' \|\sigma\|_{H^4_{\zeta,\mathbb{C}} \cap L^2_N}$. By Sobolev embedding theorem, it follows that $F_\sigma \in C^{N,\gamma}(\mathbb{R}, \mathbb{R})$, $\gamma < \frac{1}{2}$. Moreover since $\lim_{x\to+\infty} F_\sigma(x) = 0$, one has

$$\partial_x f_\sigma(x, y) = \partial_x \int_x^{+\infty} F_\sigma(y + z) F_\sigma(z)\, dz = -F_\sigma(x + y) F_\sigma(x) \ . \tag{2.145}$$

Consider first the case $j_1 \geq 1$. Then $j_2 \leq N$. By (2.145) it follows that

$$\partial_x^{j_1} \partial_y^{j_2} f_\sigma(x, y) = -\sum_{l=0}^{j_1-1} \binom{j_1 - 1}{l} F_\sigma^{(j_2+l)}(x + y) F_\sigma^{(j_1-1-l)}(x) \ , \tag{2.146}$$

where $F_\sigma^{(l)} \equiv \partial_x^l F_\sigma$. Thus $\partial_x^{j_1} \partial_y^{j_2} f_\sigma$ is a linear combination of terms of the form (2.148), with $b_\sigma = F_\sigma^{(j_1-1-l)}$ satisfying the assumption of Lemma 2.56 (i), thus $\partial_x^{j_1} \partial_y^{j_2} f_\sigma$, with $j_1 \geq 1$, satisfies (P).

Consider now the case $j_1 = 0$. Then $1 \leq j_2 \leq n \leq N+1$. Since $\partial_y F_\sigma(x+y+z) = \partial_z F_\sigma(x+y+z) = F'_\sigma(x + y + z)$, by integration by parts one obtains

$$\partial_y^{j_2} f_\sigma(x, y) = -F_\sigma^{(j_2-1)}(x + y) F_\sigma(x) - \int_0^{+\infty} F_\sigma^{(j_2-1)}(x + y + z) F'_\sigma(x + z) dz \ . \tag{2.147}$$

Then, by Lemma 2.56 (i) and (ii), $\partial_y^{j_2} f_\sigma$ is the sum of two terms which satisfy (P), thus it satisfies (P) as well. $\square$

Lemma 2.56. *Fix $c \in \mathbb{R}$, $N \in \mathbb{Z}_{\geq 0}$ and let $\sigma \in \mathscr{S}^{4,N}$. Let F_σ be given as in (2.76). Then the following holds true:*

(i) Let $\sigma \mapsto b_\sigma$ be real analytic as a map from $\mathscr{S}^{4,N}$ to $H^1_{x\geq c}$, satisfying $\|b_\sigma\|_{H^1_{x\geq c}} \leq K_c \|\sigma\|_{H^4_{\zeta,\mathbb{C}} \cap L^2_N}$, where $K_c > 0$ depends locally uniformly on $\sigma \in H^4_{\zeta,\mathbb{C}} \cap L^2_N$. Then for every integer k with $0 \leq k \leq N$, the function

$$\mathbf{H}_R(x, y) := F_\sigma^{(k)}(x + y)\, b_\sigma(x) \tag{2.148}$$

satisfies (P).

(ii) For every integer $0 \leq k \leq N$, the function

$$\mathbf{G}_R(x, y) = \int_0^{+\infty} F_\sigma^{(k)}(x + y + z) F'_\sigma(x + z) dz \tag{2.149}$$

satisfies (P).

(iii) Let $N \geq 1$ and let G_σ be a function satisfying (P). Then the function

$$\mathbf{F}_R(x,y) := \int_0^{+\infty} F'_\sigma(x+y+z) G_\sigma(x,z)\, dz \tag{2.150}$$

satisfies (P).

Proof. *(i)* $\mathbf{H}_R$ *satisfies* $(P1)$. Clearly $\mathbf{H}_R(x,\cdot) \in L^2_{y\geq 0}$ and by the continuity of the translations in L^2 one verifies that $\|\mathbf{H}_R(x+\epsilon,\cdot) - \mathbf{H}_R(x,\cdot)\|_{L^2_{y\geq 0}} \to 0$ as $\epsilon \to 0$, thus proving that $\mathbf{H}_R \in C^0_{x\geq c} L^2_{y\geq 0}$. We show now that $\mathbf{H}_R \in L^2_{x\geq c} L^2_{y\geq 0}$. By Lemma 2.42 $(A1)$, Proposition 2.33 and the assumption on b_σ, one has that

$$\|\mathbf{H}_R\|_{L^2_{x\geq c} L^2_{y\geq 0}} \leq C \|F_\sigma\|_{H^{N+1}} \|b_\sigma\|_{L^2_{x\geq c}} \leq K_c \|\sigma\|^2_{H^4_{\zeta,\mathbb{C}} \cap L^2_N} \ , \tag{2.151}$$

where $K_c > 0$ can be chosen locally uniformly for $\sigma \in H^4_{\zeta,\mathbb{C}} \cap L^2_N$.
For $0 \leq k \leq N$, $F^{(k)}_\sigma \in C^0(\mathbb{R},\mathbb{R})$ by the Sobolev embedding theorem. Thus $\mathbf{H}_R \in C^0_{x\geq c, y\geq 0}$.
Finally we show that $\mathbf{H}_R(\cdot,0) \in L^2_{x\geq c}$. We evaluate the r.h.s. of formula (2.148) at $y=0$, getting

$$\mathbf{H}_R(x,0) = F^{(k)}_\sigma(x) b_\sigma(x) \ .$$

It follows that there exists $C > 0$ and $K_c > 0$, depending locally uniformly on $\sigma \in H^4_{\zeta,\mathbb{C}} \cap L^2_N$, such that

$$\|\mathbf{H}_R(\cdot,0)\|_{L^2_{x\geq c}} \leq C \|F_\sigma\|_{H^{N+1}} \|b_\sigma\|_{H^1_{x\geq c}} \leq K_c \|\sigma\|^2_{H^4_{\zeta,\mathbb{C}} \cap L^2_N} \ , \tag{2.152}$$

where we used that both $F^{(k)}_\sigma$ and b_σ are in $H^1_{x\geq c}$.

$\mathbf{H}_R$ *satisfies* $(P2)$. It follows from (2.151) and (2.152).

$\mathbf{H}_R$ *satisfies* $(P3)$. The real analyticity property follows from Lemma 2.42 and Proposition 2.33, since for every $0 \leq k \leq N$, $\mathbf{H}_R$ is product of real analytic maps.

(ii) $\mathbf{G}_R$ *satisfies* $(P1)$. We show that $\mathbf{G}_R \in L^2_{x\geq c} L^2_{y\geq 0}$. By Lemma 2.42 $(A5)$ and Proposition 2.33 it follows that

$$\|\mathbf{G}_R\|_{L^2_{x\geq c} L^2_{y\geq 0}} \leq \|F_\sigma\|_{H^{N+1}} \|F'_\sigma\|_{L^1} \leq K_c \|\sigma\|^2_{H^4_{\zeta,\mathbb{C}} \cap L^2_N} \ , \tag{2.153}$$

where $K_c > 0$ depends locally uniformly on $\sigma \in H^4_{\zeta,\mathbb{C}} \cap L^2_N$. One verifies easily that $\mathbf{G}_R \in C^0_{x\geq c} L^2_{y\geq 0}$. In order to prove that $\mathbf{G}_R \in C^0_{x\geq c, y\geq 0}$, proceed as in Lemma 2.50.
Now we show that $\mathbf{G}_R(\cdot,0) \in L^2_{x\geq c}$. We evaluate formula (2.149) at $y=0$ getting that

$$\mathbf{G}_R(x,0) = \int_0^\infty F^{(k)}_\sigma(x+z) F'_\sigma(x+z) dz \ .$$

Let $h'_\sigma(x,z) := F'_\sigma(x+z)$. By Lemma 2.42 $(A0)$ and Proposition 2.33 one has

$$\|h'_\sigma\|_{L^2_{x\geq c} L^2_{y\geq 0}} \leq \left\| \langle x \rangle^{1/2} F'_\sigma \right\|_{L^2_{x\geq c}} \leq K_c \|\sigma\|_{H^4_{\zeta,\mathbb{C}} \cap L^2_N} \ ,$$

where $K_c > 0$ can be chosen locally uniformly for $\sigma \in H^4_{\zeta,\mathbb{C}} \cap L^2_N$. Thus by Lemma 2.42 $(A2)$ one gets

$$\|\mathbf{G}_R(\cdot,0)\|_{L^2_{x\geq c}} \leq K_c \left\|F_\sigma^{(k)}\right\|_{L^2_{x\geq c}} \|h'_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}} \leq K_c \|\sigma\|^2_{H^4_{\zeta,\mathbb{C}}\cap L^2_N} \ , \tag{2.154}$$

where $K_c > 0$ can be chosen locally uniformly for $\sigma \in H^4_{\zeta,\mathbb{C}} \cap L^2_N$.

$\mathbf{G}_R$ *satisfies* $(P2)$. It follows from (2.153) and (2.154).

$\mathbf{G}_R$ *satisfies* $(P3)$. The real analyticity property follows from Lemma 2.42 and Proposition 2.33, since for every $0 \leq k \leq N$, $\mathbf{G}_R$ is composition of real analytic maps.

(iii) $\mathbf{F}_R$ *satisfies* $(P1)$. By Lemma 2.52 (i), $\mathbf{F}_R \in C^0_{x\geq c}L^2_{y\geq 0} \cap L^2_{x\geq c}L^2_{y\geq 0}$ and

$$\|\mathbf{F}_R\|_{L^2_{x\geq c}L^2_{y\geq 0}} \leq \|F'_\sigma\|_{L^2_4} \|G_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}} \leq K_c \|\sigma\|^2_{H^4_{\zeta,\mathbb{C}}\cap L^2_N} \ .$$

Proceeding as in the proof of Lemma 2.55 $(P1)$ one shows that $\mathbf{F}_R \in C^0_{x\geq c, y\geq 0}$. Since $F'_\sigma \in H^N$, $N \geq 1$, F'_σ is a continuous function. Thus we can evaluate $\mathbf{F}_R$ at $y = 0$, obtaining $\mathbf{F}_R(x,0) = \int_0^{+\infty} F'_\sigma(x+z)G_\sigma(x,z)\,dz$. By Lemma 2.42 $(A2)$ we have that

$$\|\mathbf{F}_R(\cdot,0)\|_{L^2_{x\geq c}} \leq \|F'_\sigma\|_{L^2_{x\geq c}} \|G_\sigma\|_{L^2_{x\geq c}L^2_{y\geq 0}} \leq K_c \|\sigma\|^2_{H^4_{\zeta,\mathbb{C}}\cap L^2_N} \ .$$

The proof that $\mathbf{F}_R$ satisfies $(P2)$ and $(P3)$ follows as in the previous items. We omit the details. $\square$

Lemma 2.57. *Let $N \geq 1$ be fixed. For every $j_1, j_2 \geq 0$ with $1 \leq j_1 + j_2 \leq N$, the function $f_\sigma^{j_1,j_2}$ defined in* (2.99) *and its derivatives $\partial_y f_\sigma^{j_1,j_2}$, $\partial_x f_\sigma^{j_1,j_2}$ satisfy* (P).

Proof. First note that by Lemma 2.55 the terms $\partial_x^{j_1}\partial_y^{j_2} f_\sigma$ and its derivatives $\partial_x^{j_1+1}\partial_y^{j_2} f_\sigma$, $\partial_x^{j_1}\partial_y^{j_2+1} f_\sigma$ satisfy (P). It thus remains to show that

$$\mathbf{F}_R^{k_1,k_2}(x,y) := \int_0^{+\infty} \partial_x^{k_1} F_\sigma(x+y+z)\,\partial_x^{k_2} B_\sigma(x,z)\,dz \ , \qquad k_1 \geq 1,\ k_2 \geq 0, \quad k_1+k_2 = n \leq N \tag{2.155}$$

and its derivatives $\partial_y \mathbf{F}_R^{k_1,k_2}$, $\partial_x \mathbf{F}_R^{k_1,k_2}$ satisfy (P). Remark that, by the induction assumption in the proof of Lemma 2.38, for every integers $k_1, k_2 \geq 0$ with $k_1 + k_2 \leq n$, $\partial_x^{k_1}\partial_y^{k_2} B_\sigma$ satisfies (P).

$\mathbf{F}_R^{k_1,k_2}$ *satisfies* (P). If $k_1 = 1$, it follows by Lemma 2.56 (iii). Let $k_1 > 1$. By integration by parts $k_1 - 1$ times we obtain

$$\begin{aligned}\mathbf{F}_R^{k_1,k_2}(x,y) &= \sum_{l=1}^{k_1-1} (-1)^l \partial_x^{k_1-l} F_\sigma(x+y)(\partial_x^{k_2}\partial_z^{l-1} B_\sigma)(x,0) \\ &\quad + (-1)^{k_1-1} \int_0^{+\infty} F'_\sigma(x+y+z)\,\partial_x^{k_2}\partial_z^{k_1-1} B_\sigma(x,z)\,dz \ ,\end{aligned} \tag{2.156}$$

where we used that for $1 \leq l \leq k_1 - 1$ one has $F_\sigma^{(k_1-l)} \in H^1$ $[(\partial_x^{k_2}\partial_y^{l-1}B_\sigma)(x,\cdot) \in H^1_{y\geq 0}]$, thus $\lim_{x\to\infty} F_\sigma^{(k_1-l)}(x) = 0$ $[\lim_{y\to\infty} \partial_x^{k_2}\partial_y^{l-1}B_\sigma)(x,y) = 0]$. Consider the r.h.s. of (2.156). It is a linear combinations of terms of the form (2.148) and (2.150). By the induction assumption, these terms satisfy the hypothesis of Lemma 2.56 (i) and (iii). It follows that $\mathbf{F}_R^{k_1,k_2}$ satisfies (P), and in particular there exists a constant $K_c > 0$, depending locally uniformly on $\sigma \in H^4_{\zeta,\mathbb{C}} \cap L^2_N$, such that

$$\left\| \mathbf{F}_R^{k_1,k_2} \right\|_{L^2_{x\geq c} L^2_{y \geq 0}} + \left\| \mathbf{F}_R^{k_1,k_2}(\cdot, 0) \right\|_{L^2_{x\geq c}} \leq K_c \left\| \sigma \right\|^2_{H^4_{\zeta,\mathbb{C}} \cap L^2_N} \ . \tag{2.157}$$

$\partial_y \mathbf{F}_R^{k_1,k_2}$ *satisfies* (P). For $\epsilon \neq 0$ sufficiently small, by integration by parts k_1-times we obtain

$$\begin{aligned} \frac{\mathbf{F}_R^{k_1,k_2}(x, y+\epsilon) - \mathbf{F}_R^{k_1,k_2}(x,y)}{\epsilon} &= \sum_{l=1}^{k_1} (-1)^l \, \frac{\partial_x^{k_1-l} F_\sigma(x+y+\epsilon) - \partial_x^{k_1-l} F_\sigma(x+y)}{\epsilon} (\partial_x^{k_2} \partial_z^{l-1} B_\sigma)(x,0) \\ &\quad + (-1)^{k_1} \int_0^{+\infty} \frac{F_\sigma(x+y+\epsilon+z) - F_\sigma(x+y+z)}{\epsilon} \, \partial_x^{k_2} \partial_z^{k_1} B_\sigma(x,z) \, dz \ , \end{aligned}$$

where once again we used that for $1 \leq l \leq k_1$ one has $F_\sigma^{(k_1-l)} \in H^1$ $[(\partial_x^{k_2} \partial_y^{l-1} B_\sigma)(x, \cdot) \in H^1_{y \geq 0}]$, thus $\lim_{x\to\infty} F_\sigma^{(k_1-l)}(x) = 0$ $[\lim_{y\to\infty} \partial_x^{k_2} \partial_y^{l-1} B_\sigma)(x,y) = 0]$. Define also

$$\begin{aligned} \partial_y \mathbf{F}_R^{k_1,k_2}(x,y) := &\sum_{l=1}^{k_1} (-1)^l \, \partial_x^{k_1-l+1} F_\sigma(x+y) \, (\partial_x^{k_2} \partial_z^{l-1} B_\sigma)(x,0) \\ &+ (-1)^{k_1} \int_0^{+\infty} F'_\sigma(x+y+z) \, \partial_x^{k_2} \partial_z^{k_1} B_\sigma(x,z) \, dz \ . \end{aligned} \tag{2.158}$$

Consider the r.h.s. of equation (2.158). It is a linear combinations of terms of the form (2.148) and (2.150). By the induction assumption, these terms satisfy the hypothesis of Lemma 2.56 (i) and (iii). It follows that $\partial_y \mathbf{F}_R^{k_1,k_2}$ satisfies (P) and one has

$$\left\| \partial_y \mathbf{F}_R^{k_1,k_2} \right\|_{L^2_{x\geq c} L^2_{y \geq 0}} + \left\| \partial_y \mathbf{F}_R^{k_1,k_2}(\cdot, 0) \right\|_{L^2_{x\geq c}} \leq K'_c \left\| \sigma \right\|^2_{H^4_{\zeta,\mathbb{C}} \cap L^2_N} \tag{2.159}$$

for some constant $K'_c > 0$, depending locally uniformly on $\sigma \in H^4_{\zeta,\mathbb{C}} \cap L^2_N$. Furthermore one verifies that

$$\lim_{\epsilon \to 0} \frac{\mathbf{F}_R^{k_1,k_2}(x, \cdot + \epsilon) - \mathbf{F}_R^{k_1,k_2}(x, \cdot)}{\epsilon} = \partial_y \mathbf{F}_R^{k_1,k_2}(x, \cdot) \quad \text{ in } L^2_{y \geq 0} \ .$$

$\partial_x \mathbf{F}_R^{k_1,k_2}$ *satisfies* (P). The proof is similar to the previous case, and the details are omitted. This conclude the proof of the inductive step. □

E Hilbert transform

Define $\mathcal{H} : L^2(\mathbb{R}, \mathbb{C}) \to L^2(\mathbb{R}, \mathbb{C})$ as the Fourier multiplier operator

$$\widehat{(\mathcal{H}(v))}(\xi) = -i \, \mathrm{sign}(\xi) \ \hat{v}(\xi) \ .$$

Thus $\mathcal{H}$ is an isometry on $L^2(\mathbb{R},\mathbb{C})$. It is easy to see that $\mathcal{H}|_{H^N_{\mathbb{C}}} : H^N_{\mathbb{C}} \to H^N_{\mathbb{C}}$ is an isometry for any $N \geq 1$ – cf. [Duo01]. In case $v \in C^1(\mathbb{R},\mathbb{C})$ with $\|v'\|_{L^\infty}, \|xv(x)\|_{L^\infty} < \infty$, one has

$$\mathcal{H}(v)(k) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|k'-k|\geq \epsilon} \frac{v(k')}{k'-k} dk'$$

and obtains the estimate $|\mathcal{H}(v)(k)| \leq C(\|v'\|_\infty + \|xv(x)\|_\infty)$, where $C > 0$ is a constant independent of v and k.

Let $g \in C^1(\mathbb{R},\mathbb{R})$ with $\|g'\|_{L^\infty}, \|xg(x)\|_{L^\infty} < \infty$. Then define for $z \in \mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the function

$$f(z) := \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{g(s)}{s-z} ds \ .$$

Decompose $\frac{1}{s-z}$ into real and imaginary part

$$\frac{1}{s-z} = \frac{1}{s-a-ib} = \frac{s-a}{(s-a)^2+b^2} + i\frac{b}{(s-a)^2+b^2}$$

to get the formulas for the real and imaginary part of $f(z)$

$$\mathrm{Re}\, f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{b}{(s-a)^2+b^2} g(s) ds \ , \tag{2.160}$$

$$\mathrm{Im}\, f(z) = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{s-a}{(s-a)^2+b^2} g(s) ds \ . \tag{2.161}$$

The following Lemma is well known and can be found in [Duo01].

Lemma 2.58. *The function f is analytic and admits a continuous extension to the real line. Furthermore it has the following properties for any $a \in \mathbb{R}$:*

(i) $\lim_{b\to 0^+} \mathrm{Im}\, f(a+bi) = \mathcal{H}(g)(a)$.

(ii) $\lim_{b\to 0^+} \mathrm{Re}\, f(a+bi) = g(a)$.

(iii) There exists $C > 0$ such that $|f(z)| \leq \frac{C}{1+|z|}$, $\forall\, z \in \{z : \mathrm{Im}\, z \geq 0\}$.

(iv) Let $\tilde{f}(z)$ be a continuous function on $\mathrm{Im}\, z \geq 0$ which is analytic on $\mathrm{Im}\, z > 0$ and satisfies $\mathrm{Re}\, \tilde{f}_{|\mathbb{R}} = g$ and $|\tilde{f}(z)| = O(\frac{1}{|z|})$ as $|z| \to \infty$, then $\tilde{f} = f$.

The next lemma follows from the commutator estimates due to Calderón [Cal65]:

Lemma 2.59 ([Cal65])**.** *Let $b : \mathbb{R} \to \mathbb{R}$ have first-order derivative in L^∞. For any $p \in (1,\infty)$ there exists $C > 0$, such that*

$$\|[\mathcal{H}, b]\, \partial_x g\|_{L^p} \leq C\, \|g\|_{L^p} \ .$$

We apply this lemma to prove the following result:

Lemma 2.60. *Let $M \in \mathbb{Z}_{\geq 1}$ be fixed. Then $\mathcal{H} : H^M_{\zeta,\mathbb{C}} \to H^M_{\zeta,\mathbb{C}}$ is a bounded linear operator.*

Proof. Let $f \in H^M_{\zeta,\mathbb{C}}$. As the Hilbert transform commutes with the derivatives, we have that $\mathcal{H}(f) \in H^{M-1}_{\mathbb{C}}$. Next we show that if $\zeta\partial_k^M f \in L^2$, then $\zeta\partial_k^M \mathcal{H}(f) = \zeta\mathcal{H}(\partial_k^M f) \in L^2$. By Lemma 2.59 with $p = 2$, $g = \partial_k^{M-1} f$ and $b = \zeta$, we have that

$$\left\|\zeta\mathcal{H}(\partial_k^M f)\right\|_{L^2} \leq \left\|\mathcal{H}(\zeta\partial_k^M f)\right\|_{L^2} + \left\|[\mathcal{H},\zeta]\,\partial_k^M f\right\|_{L^2} \leq \|f\|_{H^M_{\zeta,\mathbb{C}}} + C\left\|\partial_k^{M-1} f\right\|_{L^2} < \infty\,.$$

$\square$

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